On

Positive TAGED

with a

Bounded Number of Constraints

P.-C. Héam, Vincent Hugot, O. Kouchnarenko

{pheam,vhugot,okouchna}@femto-st.fr

University of Franche-Comté

DGA & INRIA/CASSIS & FEMTO-ST (DISC)

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Tree Automata

[Comon et al., 2008a]

Introduced in the fifties; **regular tree languages**:

- model-checking: programs, protocols,. . .
- automated theorem-proving
- XML schema and (esp. variants) query languages
- . . . and so much more

Doesn't deal with **comparisons** and **non-linearity**:

- $\{ f(u, u) \mid u \in \mathcal{T}(\Sigma) \}$ e.g. password verification
- $\{ f(u, v) \mid u, v \in \mathcal{T}(\Sigma), u \neq v \}$ e.g. primary keys
- $\mathcal{R}(\ell)$, ℓ regular, $\mathcal{R}$ a TRS e.g. $\{ g(x) \to f(x, x) \}(\mathcal{T}(\mathbb{A}))$

Tree Automata

Bottom-Up, Non-Deterministic, Finite

Tree Automaton $\mathcal{A} = \langle \mathbb{A}, Q, F, \Delta \rangle$:

$\mathbb{A}$	finite **ranked alphabet**
Q	finite set of **states**
F	**final** states, $F \subseteq Q$
Δ	finite set of **transitions**

Transition $r \in \Delta$:

$$\sigma(q_1, \dots, q_n) \to q \qquad \sigma \in \mathbb{A}_n \quad q_1, \dots, q_n, q \in Q$$

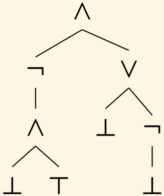
Tree Automata

Bottom-Up, Non-Deterministic, Finite

$\mathbb{A} = \{\, \wedge, \vee/2, \neg/1, \top, \bot/0 \,\}$, $Q = \{\, q_0, q_1 \,\}$, $F = \{q_1\}$, $\Delta =$

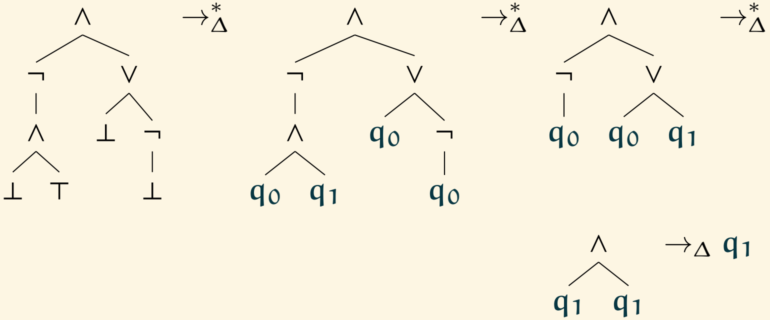
$$\left\{ \begin{array}{l} \top \to q_1, \quad \bot \to q_0, \quad \neg(q_b) \to q_{\neg b} \\ \wedge(q_b, q_{b'}) \to q_{b \wedge b'}, \quad \vee(q_b, q_{b'}) \to q_{b \vee b'} \end{array} \;\middle|\; b, b' \in \{\, 0, 1 \,\} \right\}$$

$t =$



Tree Automata

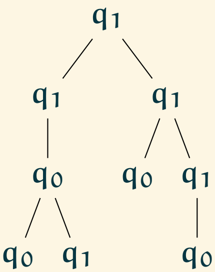
Bottom-Up, Non-Deterministic, Finite



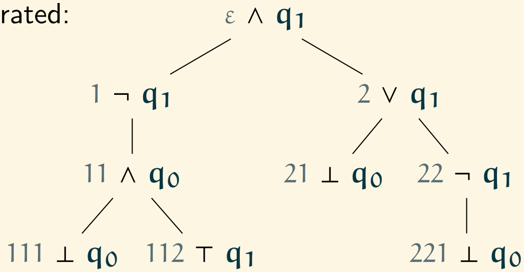
Tree Automata

Runs and Languages

The reduction $t \to_{\Delta}^{*} q_1$ is captured by the **run**:



decorated:



Tree Automata

With Global Equality Constraints [Filiot et al., 2008]

TAGE, $TA^=$, Positive TAGED, $\mathcal{A} = \langle \mathbb{A}, Q, F, \Delta, \approxeq \rangle$:

$\langle \mathbb{A}, Q, F, \Delta \rangle$	vanilla **tree automaton** $ta(\mathcal{A})$
$\approxeq$	equality **constraints**, $\approxeq \subseteq Q^2$

Constraint $p \approxeq q$:

run ρ of $\mathcal{A}$ on t:

- **run** of $ta(\mathcal{A})$ on t
- **satisfying** $\approxeq$: $\forall \alpha, \beta \in \mathcal{P}(t);\ \rho(\alpha) \approxeq \rho(\beta) \Rightarrow t|_\alpha = t|_\beta$

accepting run: accepting for $ta(\mathcal{A})$

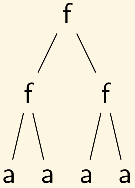
Tree Automata

With Global Equality Constraints [Filiot et al., 2008]

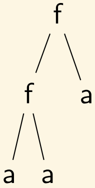
$\mathbb{A} = \{ a/_0, f/_2 \}$, $Q = \{ q, \hat{q}, q_f \}$, $F = \{ q_f \}$, $\hat{q} \approxeq \hat{q}$, and

$$\Delta = \{ f(\hat{q}, \hat{q}) \to q_f,\ f(q,q) \to q,\ f(q,q) \to \hat{q},\ a \to q,\ a \to \hat{q} \}$$

$u =$



and $v =$



Tree Automata

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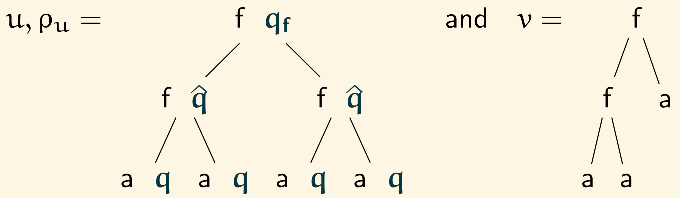

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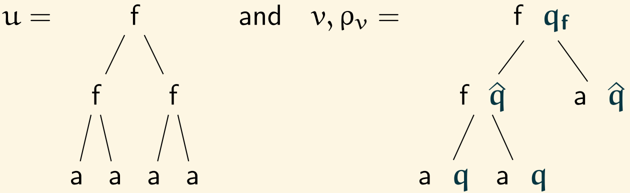
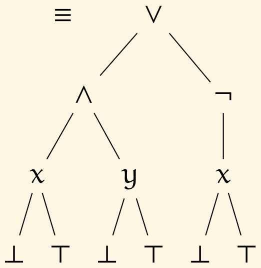


Tree Automata

With Global Equality Constraints [Filiot et al., 2008]

$\mathbb{A} = \{\wedge, \vee/_2, \neg/_1, \top, \bot/_0\} \uplus \mathbb{X}$, $Q = \{q_0, q_1\} \uplus \{v_x \mid x \in \mathbb{X}\}$ and $F = \{q_1\}$, new rules $\top \to v_x$, $\bot \to v_x$, $x(q_0, v_x) \to q_1$, $x(v_x, q_1) \to q_0$ for each $x \in \mathbb{X}$, $v_x \approxeq v_x$.

$(x \wedge y) \vee \neg x \quad \equiv$



Tree Automata

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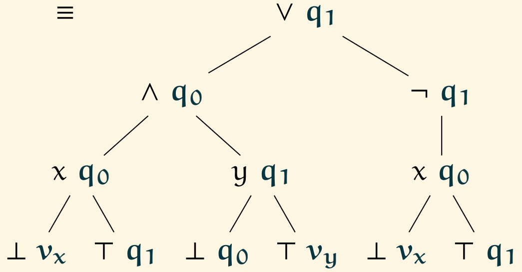


TA versus RTA versus $TA^=$

Closure, Complexity and Decidability

	TA	RTA ($p \approx p$)	$TA^=$
$\cup$	PTime	PTime	PTime
$\cap$	PTime	ExpTime	ExpTime
$\neg$	ExpTime	$\varnothing$	$\varnothing$
$t \in \mathcal{L}(\mathcal{A})$?	PTime	NP-c	NP-c [a]
$\mathcal{L}(\mathcal{A}) = \varnothing$?	linear-time	linear-time	ExpTime-c
$\lvert\mathcal{L}(\mathcal{A})\rvert \in \mathbb{N}$?	PTime	PTime	ExpTime-c
$\mathcal{L}(\mathcal{A}) = \mathcal{T}(\Sigma)$?	ExpTime-c	*undecidable*	*undecidable*
$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$?	ExpTime-c	*undecidable*	*undecidable*
$\mathcal{L}(\bigcap_i \mathcal{A}_i) = \varnothing$?	ExpTime-c	ExpTime-c	ExpTime-c

[a] SAT solver approach: [Héam et al., 2010].

TA versus RTA versus $TA^=$

Closure, Complexity and Decidability

	TA	**RTA** $(p \approx p)$	$\mathbf{TA^=}$
$\cup$	PTIME	PTIME	PTIME
$\cap$	PTIME	EXPTIME	EXPTIME
$\neg$	EXPTIME	$\varnothing$	$\varnothing$
$t \in \mathcal{L}(\mathcal{A})$?	PTIME	NP-c	NP-c [(a)]
$\mathcal{L}(\mathcal{A}) = \varnothing$?	linear-time	**linear-time**	**ExpTime-c**
$\lvert\mathcal{L}(\mathcal{A})\rvert \in \mathbb{N}$?	PTIME	**PTime**	**ExpTime-c**
$\mathcal{L}(\mathcal{A}) = \mathcal{T}(\Sigma)$?	EXPTIME-c	*undecidable*	*undecidable*
$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$?	EXPTIME-c	*undecidable*	*undecidable*
$\mathcal{L}(\bigcap_i \mathcal{A}_i) = \varnothing$?	EXPTIME-c	EXPTIME-c	EXPTIME-c

(a) SAT solver approach: [Héam et al., 2010].

$TA^=$ versus $TA_k^=$

Restriction on the **kind** of constraints ⇒ **lower complexity** (RTA)

Restriction on the **number** of constraints ⇒ **?**

$TA_k^=$ $\mathcal{A} = \langle \Sigma, Q, F, \Delta, \approxeq \rangle$:

$\langle \Sigma, Q, F, \Delta, \approxeq \rangle$	$TA^=$ $\mathcal{A}$
$\approxeq$	such that $\mathrm{Card}(\approxeq) \leqslant k$

$\mathsf{TA}^{=}$ versus $\mathsf{TA}^{=}_{k}$

Restriction on the **kind** of constraints $\Rightarrow$ **lower complexity** (RTA)

Restriction on the **number** of constraints $\Rightarrow$ **?**

$\mathsf{TA}^{=}_{k}$ $\mathcal{A} = \langle \Sigma, Q, F, \Delta, \approxeq \rangle$:

$\langle \Sigma, Q, F, \Delta, \approxeq \rangle$	$\mathsf{TA}^{=}$ $\mathcal{A}$
$\approxeq$	such that $\operatorname{Card}(\approxeq) \leqslant k$

Expressive power?

$$\mathsf{TA} = \mathsf{TA}^{=}_{0} \subset \mathsf{TA}^{=}_{1} \subset \cdots \subset \mathsf{TA}^{=}_{k} \subset \mathsf{TA}^{=}_{k+1} \subset \cdots \subset \mathsf{TA}^{=} = \bigcup_{k \in \mathbb{N}} \mathsf{TA}^{=}_{k}$$

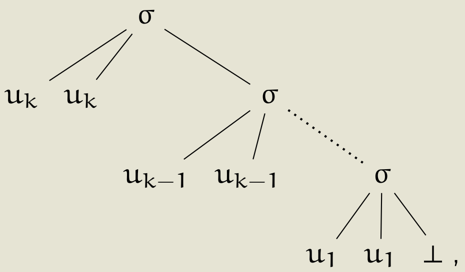
so $\forall k \geqslant 0,\ \mathcal{L}(\mathsf{TA}^{=}_{k}) \subseteq \mathcal{L}\left(\mathsf{TA}^{=}_{k+1}\right)$. Are the inclusions **strict**? Up to some rank? Is there a $k \in \mathbb{N}$ such that $\mathcal{L}(\mathsf{TA}^{=}_{k}) = \mathcal{L}(\mathsf{TA}^{=})$?

Expressive Power

The Separation Languages $L = (\ell_k)_{k \in \mathbb{N}}$ [Hugot, 2013]

$$\biguplus_{i=1}^{k} \mathbb{A}_i \uplus \{\, \sigma/3, \bot/0 \,\} \qquad \mathbb{A}_i = \{\, a_i, b_i/0, f_i, g_i/2 \,\}$$

$$\ell_0 = \{\bot\} \quad \forall k \geqslant 1, \ell_k = \{\, \sigma(u, u, t_{k-1}) \mid u \in \mathcal{T}(\mathbb{A}_k), t_{k-1} \in \ell_{k-1} \,\}$$



$u_i \in \mathcal{T}(\mathbb{A}_i)\ \forall i$

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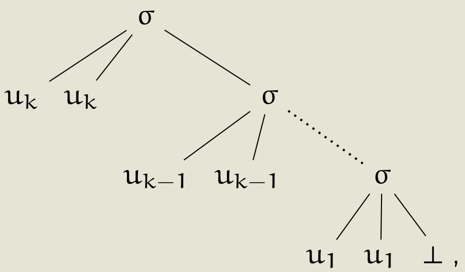

$u_i \in \mathcal{T}(\mathbb{A}_i)\ \forall i$

$\ell_1 \in \mathcal{L}(\mathsf{TA}_1^{=}) \setminus \mathcal{L}(\mathsf{TA})$ $\approx$ ground instances of $f(x,x)$.

$\ell_k \in \mathcal{L}(\mathsf{TA}_k^{=}) \setminus \mathcal{L}(\mathsf{TA}_{k-1}^{=}), \quad \forall k \geqslant 1.$

Expressive Power

Show $\ell_k \in \mathcal{L}(\mathsf{TA}_k^{\equiv}) \setminus \mathcal{L}(\mathsf{TA}_{k-1}^{\equiv})$ [Hugot, 2013]

Show $\ell_k \in \mathcal{L}(\mathbf{TA}_k^{\equiv})$: $\mathcal{A}_k \in \mathsf{TA}_k^{\equiv}$ such that $\mathcal{L}(\mathcal{A}_k) = \ell_k$

$\mathcal{U}_i \in \mathsf{TA}$ universal, $\mathcal{U}_i{:}F = \{q_i^u\}$, for all i. $\mathcal{A}_k$ is

$$Q = \{q_0^v\} \uplus \biguplus_{i=1}^{k} \mathcal{U}_i{:}Q \uplus \{q_i^v\} \qquad F = \{q_1^v\} \qquad q_i^u \approxeq q_i^u,\ \forall i \in [\![1,k]\!]$$

$$\Delta = \left\{ \sigma(q_i^u, q_i^u, q_{i-1}^v) \to q_i^v \;\middle|\; i \in [\![1,k]\!] \right\} \cup \{ \bot \to q_0^v \}.$$

Expressive Power

Show $\ell_k \in \mathcal{L}(\mathsf{TA}^{\equiv}_{k}) \setminus \mathcal{L}(\mathsf{TA}^{\equiv}_{k-1})$ [Hugot, 2013]

Show $\ell_k \notin \mathcal{L}(\mathbf{TA}^{\equiv}_{k-1})$:

active constrained states:

$$\operatorname{acs}\rho = \{\, \rho(\alpha) \mid \alpha \in \mathcal{P}(\rho), \exists \beta \in \mathcal{P}(\rho) \setminus \{\alpha\} : \rho(\alpha) \approxeq \rho(\beta) \,\}$$

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Show $\ell_k \in \mathcal{L}(\mathsf{TA}^{\equiv}_{k}) \setminus \mathcal{L}(\mathsf{TA}^{\equiv}_{k-1})$ [Hugot, 2013]

Show $\ell_k \notin \mathcal{L}(\mathbf{TA}^{\equiv}_{k-1})$:

- Assume $\ell_k \in \mathcal{L}(\mathsf{TA}^{\equiv}_{k-1})$ i.e. $\exists \mathcal{A} \in \mathsf{TA}^{\equiv}_{k-1} : \mathcal{L}(\mathcal{A}) = \ell_k$

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- $\forall \rho,\ \nexists \alpha, \beta : \alpha \neq \beta, \alpha \in 3^*, \rho(\alpha) \approxeq \rho(\beta)$

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- Pick $t \in \ell_k$ such that $\left|t|_\alpha\right| > |Q|$, for all $\alpha \in 3^*(1+2)$

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- Pick $t \in \ell_k$ such that $\left| t|_\alpha \right| > |Q|$, for all $\alpha \in 3^*(1+2)$
- Suppose $\exists \alpha \in 3^*(1+2)$ such that $\operatorname{ran} \rho|_\alpha \cap \operatorname{acs} \rho = \varnothing$. $\mathcal{A}$ acts as BUTA wrt. $t|_\alpha$; pump $\rho|_\alpha$, get $t' \notin \ell_k$, but $t' \in \mathcal{L}(\mathcal{A})$.

Expressive Power

Show $\ell_k \in \mathcal{L}(\mathrm{TA}^{=}_{k}) \setminus \mathcal{L}(\mathrm{TA}^{=}_{k-1})$ [Hugot, 2013]

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- $\forall \alpha \in 3^*(1+2),\ \operatorname{ran} \rho|_\alpha \cap \operatorname{acs} \rho \neq \varnothing$

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- Pick $t \in \ell_k$ such that $\left| t|_\alpha \right| > |Q|$, for all $\alpha \in 3^*(1+2)$
- $\forall \alpha \in 3^*(1+2),\ \operatorname{ran} \rho|_\alpha \cap \operatorname{acs} \rho \neq \varnothing$
- $i \neq j$, p_i acs for u_i, p_j for u_j. $\exists$ acs $q_i, q_j : p_i \approxeq q_i, p_j \approxeq q_j$. Suppose q_i in subrun of u_j. Then $\exists s_i \trianglelefteq u_i, s_j \trianglelefteq u_j$, $s_i = s_j$. But $u_i \in \mathcal{T}(\mathbb{A}_i)$ and $u_j \in \mathcal{T}(\mathbb{A}_j)$, thus $s_i \in \mathcal{T}(\mathbb{A}_i)$ and $s_j \in \mathcal{T}(\mathbb{A}_j)$. $\mathcal{T}(\mathbb{A}_i) \cap \mathcal{T}(\mathbb{A}_j) = \varnothing$, thus $s_i = s_j \in \varnothing$.

Expressive Power

Show $\ell_k \in \mathcal{L}(\mathrm{TA}^{\equiv}_{k}) \setminus \mathcal{L}(\mathrm{TA}^{\equiv}_{k-1})$ [Hugot, 2013]

Show $\ell_k \notin \mathcal{L}(\mathbf{TA}^{\equiv}_{k-1})$:

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- Pick $t \in \ell_k$ such that $\left| t|_\alpha \right| > |Q|$, for all $\alpha \in 3^*(1+2)$
- $\forall \alpha \in 3^*(1+2),\ \operatorname{ran} \rho|_\alpha \cap \operatorname{acs} \rho \neq \varnothing$
- Each pair of u_i needs its own fresh state(s) $p_i \approxeq q_i$

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Show $\ell_k \in \mathcal{L}(\mathrm{TA}^{=}_{k}) \setminus \mathcal{L}(\mathrm{TA}^{=}_{k-1})$ [Hugot, 2013]

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- Each pair of u_i needs its own fresh state(s) $p_i \approxeq q_i$
- $\mathcal{A}$ does not exist, contradiction.

The Membership Problem

General Idea & Strategy

Membership complexity : $t \in \mathcal{L}(\mathcal{A})$?

NP-complete	for	$TA^{=}$
PTime	for	$TA^{=}_{k}, \forall k \in \mathbb{N}$

Proof **Strategy** :

- Choose each $P \subseteq \mathrm{dom} \approxeq = \{ p \mid \exists q : p \approxeq q \text{ or } q \approxeq p \}$
- Given P, turn $\approxeq$ into an equivalence relation $\asymp_P$
- Try all possible "housings" of the $\approxeq$-classes into t
- For each housing, try to build an accepting run

$\approxeq$ is Not an Equivalence

(but we can pretend it is)

Example: Given $p \approxeq r$ and $r \approxeq q$, what of $p \approxeq q$?

Does r actually appear in the run ?

yes : $p \approxeq q$ implied

no : $p \approxeq r$ and $r \approxeq q$ are moot.

Fix $P \subseteq \mathrm{dom} \approxeq$. Any run ρ such that $(\mathbf{ran}\,\rho) \cap (\mathbf{dom} \approxeq) = \mathbf{P}$ is accepting for $\mathcal{A}$ iff it is so for

$$\mathcal{A}_P = \wr \mathcal{A} \mid \approxeq := \left(\approxeq \cap P^2\right)^{\equiv} \int ,$$

symmetric, transitive, reflexive closure under $\mathrm{dom}(\approxeq \cap P^2)$.

Groups & Similarity Classes

Groups $\mathbb{G}_P$: set of $\approxeq$-equivalence classes (given P)

$$\mathbb{G}_P = \frac{\mathrm{dom}(\approxeq \cap P^2)}{(\approxeq \cap P^2)^{\equiv}} = \frac{\mathrm{dom}(\approxeq \cap P^2)}{\asymp_P}$$

Similarity **Classes** $\mathbb{S}_t$ of t :

$$\forall \alpha, \beta \in \mathcal{P}(t);\ \alpha \sim \beta \iff t|_\alpha = t|_\beta$$

$$\textbf{classes } \mathbb{S}_t = \mathcal{P}(t)/_\sim$$

Housings

And Their Compatibility with the Constraints

Characterisation of **Satisfaction** of $\approxeq$:

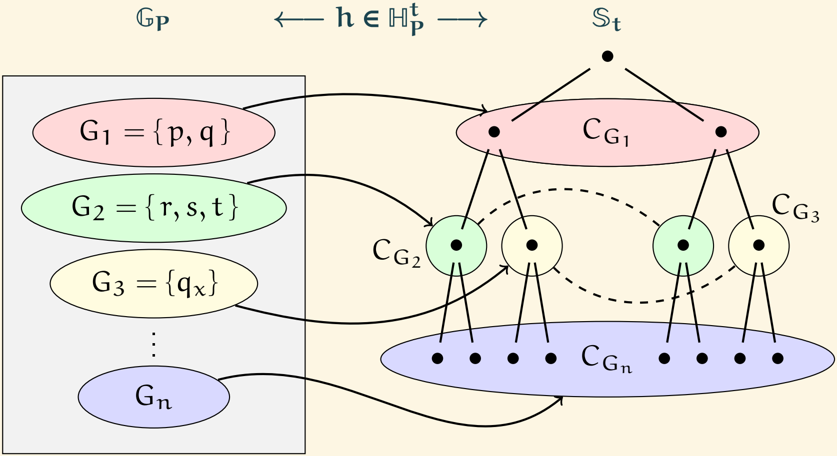
$$\forall G \in \mathbb{G}_P;\ \exists C_G \in \mathbb{S}_t : \rho^{-1}(G) \subseteq C_G$$

Housings $\mathbb{H}_P^t$ of P within t :

The map $G \mapsto C_G$ is a **P-housing of ρ in t**, **compatible** with ρ

$$\mathbb{H}_P^t = \mathbb{G}_P \to \mathbb{S}_t$$

is the set of all possible P-housings on t.



$\mathbb{G}_{\mathbf{P}}$
$\longleftarrow \mathbf{h} \in \mathbb{H}_{\mathbf{P}}^{\mathbf{t}} \longrightarrow$
$\mathbb{S}_{\mathbf{t}}$
$G_1 = \{p, q\}$
$G_2 = \{r, s, t\}$
$G_3 = \{q_x\}$
$\vdots$
G_n
C_{G_1}
C_{G_2}
C_{G_3}
C_{G_n}

Proof Outline

For $TA_k^=$

Operations Needed :

- **Choose P:** 2^{2k} possible $P \subseteq \mathrm{dom} \approxeq$
- **Choose housing:** $\left|\mathbb{S}_t{}^{\mathbb{G}_P}\right| = |\mathbb{S}_t|^{|\mathbb{G}_P|} \leqslant \|t\|^{2k}$ P-housings on t
- $\Rightarrow 4^k \cdot \|t\|^{2k}$ tests in total

↬ **polynomial** compatibility test = variant of **reachability**

Is a final state reachable if states $q \in P$ can only go in $h([q]_{\asymp_P})$?

Compatibility Test

In Polynomial Time

Simple variant of **reachability** algorithm:

Given P and $h \in \mathbb{H}_P^t$, there exists a compatible run iff

$$\Phi_t^{P,h}(\varepsilon) \cap F \neq \varnothing \ ,$$

where

$$\Phi_t^{P,h}(\alpha) = \left\{ q \in Q \;\middle|\; \begin{array}{l} t(\alpha)(p_1, \dots, p_n) \to q \in \Delta \\ \forall i \in [\![1, n]\!],\ p_i \in \Phi_t^{P,h}(\alpha.i) \\ q \in \bigcup \mathbb{G}_P \implies \alpha \in h([q]_{\asymp_P}) \\ q \notin \mathrm{dom}(\approxeq) \setminus P \end{array} \right\} .$$

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Rigidification

Problem : Given TA$^{=}$ $\mathcal{A}$, build equivalent RTA $\mathcal{B}$.

General Result [Filiot, 2008, Lem. 5.3.5]

Exponential construction: $\|\mathcal{B}\| \leqslant \mathbf{O}(2^{\|\mathcal{A}\|^2})$

In the case of $\mathbf{TA}_1^{=}$:

Polynomial construction: $\|\mathcal{B}\| \leqslant \mathbf{O}(\|\mathcal{A}\|^2)$

Idea : Simulate a constraint $p \approxeq q,\ p \neq q$ by a TA intersection

Rigidification: Construction

$$\mathcal{B} = \mathcal{B}_p^\neg \uplus \mathcal{B}_q^\neg \uplus \wr \mathcal{A} \mid Q', \Delta', q_f \approxeq q_f \int$$

$$\mathcal{B}_p^\neg = \wr \mathcal{A} \mid Q \setminus \{p\} \int \qquad \mathcal{B}_q^\neg = \wr \mathcal{A} \mid Q \setminus \{q\} \int$$

$$Q' = (Q \setminus \{p, q\}) \uplus (\mathcal{B}_{pq} : Q) \qquad \Delta' = \Delta_{pq}^{q_f} \uplus (\mathcal{B}_{pq} : \Delta)$$

$$\mathcal{B}_{pq} = \mathcal{B}_p \otimes \mathcal{B}_q \qquad q_f = (p, q)$$

$$\mathcal{B}_p = \wr \mathcal{B}_q^\neg \mid F := \{p\}, \Delta := \Delta_p \int \qquad \mathcal{B}_q = \wr \mathcal{B}_p^\neg \mid F := \{q\}, \Delta := \Delta_q \int$$

$$\Delta_p = \mathcal{B}_q^\neg : \Delta \setminus \{\ldots p \ldots \to \ldots\} \qquad \Delta_q = \mathcal{B}_p^\neg : \Delta \setminus \{\ldots q \ldots \to \ldots\}$$

$\Delta_{pq}^{q_f}$ is $\mathcal{A} : \Delta$ from which all left-hand side occurrences of p or q have been replaced by q_f.

Emptiness

Outline of the Result and Proof

Complexity of **Emptiness** : $\mathcal{L}(\mathcal{A}) = \varnothing$?

PTime (quadratic)	for	$\mathrm{TA}_1^{\equiv}$
ExpTime-complete	for	$\mathrm{TA}_k^{\equiv}$, $k \geqslant 2$

$\mathrm{TA}_1^{\equiv}$: immediate by **rigidification**. Emptiness for RTA: linear time

$\mathrm{TA}_2^{\equiv}$: Reduction of **intersection-emptiness** of n TA $\mathcal{A}_1, \dots, \mathcal{A}_n$.
Generalisation of the usual argument [Filiot et al., 2008, Thm. 1] from "unlimited constraints" to **"two constraints"**

$$L = \varnothing \iff \bigcap_{i=1}^{n} \mathcal{L}(\mathcal{A}_i) = \varnothing$$

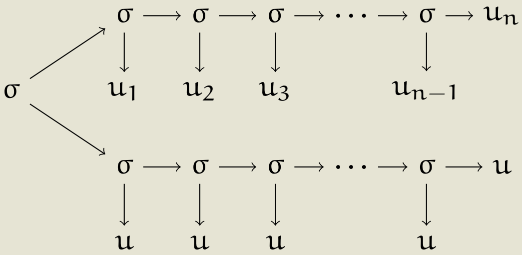

Figure: Reduction of intersection-emptiness: the language.

where $\forall i,\ x_i \in \mathcal{L}(\mathcal{A}_i)$ and $x = x_i$

Finiteness

Outline of the Result and Proof

Complexity of **Finiteness** : $|\mathcal{L}(\mathcal{A})| \in \mathbb{N}$?

PTime	for	$\mathsf{TA}_1^{\equiv}$
ExpTime-complete	for	$\mathsf{TA}_k^{\equiv}$, $k \geqslant 2$

$\mathsf{TA}_1^{\equiv}$: immediate by **rigidification**. Finiteness for RTA is PTime

$\mathsf{TA}_2^{\equiv}$: Reduction of **Emptiness for $\mathsf{TA}_2^{\equiv}$**.

Finiteness

Outline of the Result and Proof

$$\mathcal{A}' = \wr\, \mathcal{A} \mid Q \uplus \{p\}, F := \{p\}, \Sigma \uplus \{\sigma/1\}, \Delta' \int$$
$$\text{where } \Delta' = \Delta \cup \{\, \sigma(q_f) \to p \mid q_f \in F \,\} \cup \{\, \sigma(p) \to p \,\}$$

$$\text{if } \mathcal{L}(\mathcal{A}) = \varnothing \quad \textbf{then} \quad \mathcal{L}(\mathcal{A}') = \varnothing$$
$$\text{if } t \in \mathcal{L}(\mathcal{A}) \quad \textbf{then} \quad \sigma^+(t) \subseteq \mathcal{L}(\mathcal{A}')$$

$\mathcal{L}(\mathcal{A}')$ is **finite** $\iff$ $\mathcal{L}(\mathcal{A})$ is **empty**

Summary
and Perspectives

Refined **complexity** and **expressiveness** results:

- **Expressiveness:** $\mathsf{TA}_k^{\equiv}$ form a strict hierarchy
- **Membership:** NP-c for $\mathsf{TA}^{\equiv}$, but PTIME for $\mathsf{TA}_k^{\equiv}$, $\forall k$
- **Emptiness:** quadratic for $\mathsf{TA}_1^{\equiv}$, EXPTIME-complete for $\mathsf{TA}_2^{\equiv}$
- **Finiteness:** PTIME for $\mathsf{TA}_1^{\equiv}$, EXPTIME-complete for $\mathsf{TA}_2^{\equiv}$

Left **to do**:

Effects of $\not\approx$, flat constraints, efficient heuristics, etcetera.

Tree Automata With Constraints

Brief survey: [Hugot, 2013]

Positional constraints:

- **TALEDC** [Mongy, 1981]: local equality and disequality constraints. $f(q_1, q_2, q_3)[13 \approx 2, 12 \not\approx 13] \to q$.

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- **GRA** [Caron et al., 1994]: relaxes RA, but only where constraints between brothers are involved.

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Global constraints:

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- **Parick+E**, **NParick+EDB** [Barguñó et al., 2010]:

$$\sum_{q \in Q} a_q |q| \geqslant b \quad \text{or} \quad \sum_{q \in Q} a_q \|q\| \geqslant b \quad \text{with } a_q, b \in \mathbb{Z}, \forall q \in Q\ .$$

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- **DAGA** [Charatonik, 1999]: equivalent to TAGD.

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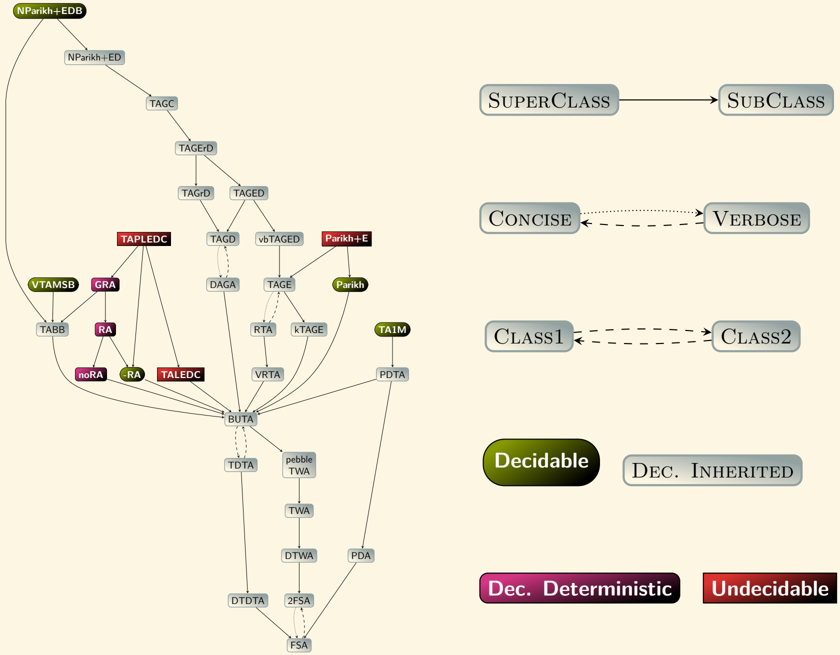
- **DAGA** [Charatonik, 1999]: equivalent to TAGD.
- **TA1M** [Comon and Cortier, 2005, Comon et al., 2008b]: one tree memory generalises pushdown automata.


NParikh+EDB
NParikh+ED
TAGC
TAGErD
TAGrD
TAGED
TAPLEDC
TAGD
vbTAGED
Parikh+E
VTAMSB
GRA
DAGA
TAGE
Parikh
TABB
RA
RTA
kTAGE
TA1M
noRA
-RA
TALEDC
VRTA
PDTA
BUTA
TDTA
pebble TWA
TWA
DTWA
PDA
DTDTA
2FSA
FSA
SuperClass
SubClass
Concise
Verbose
Class1
Class2
Decidable
Dec. Inherited
Dec. Deterministic
Undecidable

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