From Linear Temporal Logic to Rewrite Propositions

Towards a New Model-Checking Approach

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Model-Checking Process Proposal

R. Courbis, P.-C. Héam, O. Kouchnarenko in CIAA 2009, [1]

"The system $\mathcal{R}$ satisfies the property"...

$$\mathcal{R}, \Pi \models \Box(X \Rightarrow \bullet Y)$$

$\mathcal{R}$	is a Term Rewriting System (TRS)
$X, Y \subseteq \mathcal{R}$	are sets of rules
$\Pi \subseteq \mathcal{T}(\mathbb{A})$	is the initial language

Example:

$X =$ "ask PIN code" $= \{\mathsf{ask}\}$

$Y =$ "authenticate or cancel" $= \{\mathsf{auth}_1, \mathsf{auth}_2, \mathsf{can}\}$

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"The system $\mathcal{R}$ satisfies the property"...

$$\mathcal{R}, \Pi \models \ \Box(X \Rightarrow \bullet Y)$$

... is equivalent to the Rewrite Proposition (RP)...

$$[\mathcal{R} \setminus Y]\left(X\left(\mathcal{R}^*(\Pi)\right)\right) = \varnothing \ \wedge \ X\left(\mathcal{R}^*(\Pi)\right) \subseteq Y^{-1}(\mathcal{T}(\mathbb{A}))$$

... semi-decided by TAGED-based procedure

$\texttt{IsEmpty}(\texttt{OneStep}(\mathcal{R} \setminus Y, \texttt{Approx}(\mathcal{A}, \mathcal{R})), X)$ and $\texttt{Subset}(\texttt{OneStep}(X, \texttt{Approx}(\mathcal{A}, \mathcal{R})), \texttt{Backward}(Y))$, **where** $\mathfrak{Lang}(\mathcal{A}) = \Pi$, $\mathfrak{Lang}(\texttt{Approx}(\mathcal{A}, \mathcal{R})) \supseteq \mathcal{R}^*(\mathfrak{Lang}(\mathcal{A}))$ is given in [2, 3], and assuming Y is left-linear.

Our Goals...

... make it work!

1. **Generalise translation into Rewrite Propositions (RP)**
   From three specific formulæ [1] to a fragment of LTL
2. **Generalise translation from RP to TAGED semi-algos**
   At least for a fragment of possible RP
   Relatively easy...
3. **Combine them into a full (semi-)verification chain**

The present work deals with the **first step** only

Intuition: No Syntactic Translation

R. Courbis, P.-C. Héam, O. Kouchnarenko in CIAA 2009, [1]

1. $\mathcal{R}, \Pi \models \Box(X \Rightarrow \bullet Y)$

$$[\mathcal{R} \setminus Y]\left(X\left(\mathcal{R}^*(\Pi)\right)\right) = \varnothing \ \wedge \ X\left(\mathcal{R}^*(\Pi)\right) \subseteq Y^{-1}(\mathcal{T}(\mathbb{A}))$$

2. $\mathcal{R}, \Pi \models \neg Y \ \wedge \ \Box(\bullet Y \Rightarrow X)$

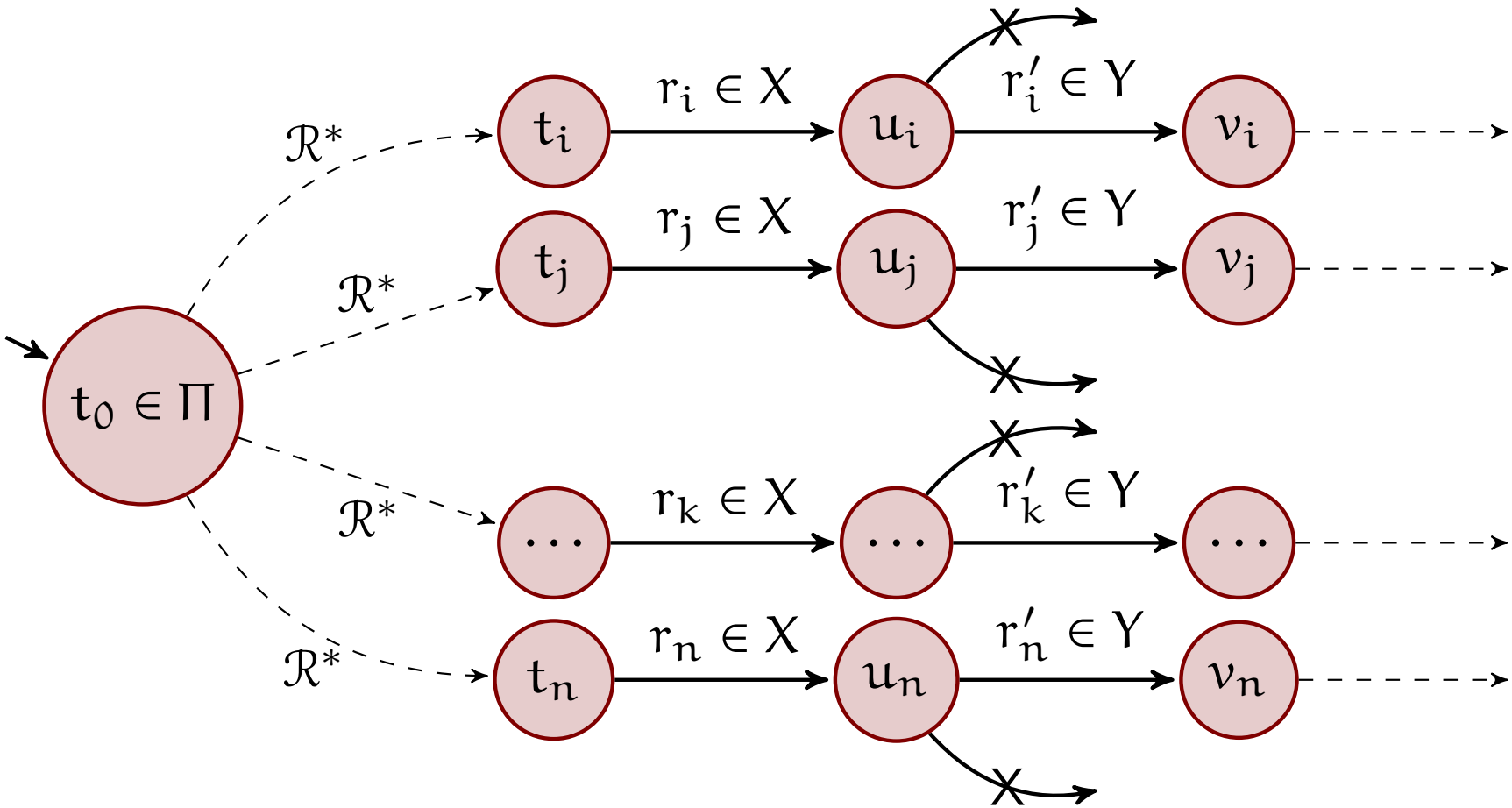
$$Y(\Pi) = \varnothing \ \wedge \ Y\left([\mathcal{R} \setminus X]\left(\mathcal{R}^*(\Pi)\right)\right) = \varnothing$$

3. $\mathcal{R}, \Pi \models \Box(X \Rightarrow \circ \Box \neg Y)$

$$Y\left(\mathcal{R}^*\left(X\left(\mathcal{R}^*(\Pi)\right)\right)\right) = \varnothing$$

Maximal Rewrite Words

Coding the Behaviour of the System: $\Box(X \Rightarrow \bullet Y)$



Maximal Rewrite Words

Coding the Behaviour of the System

Finite or Infinite Words on $\mathcal{R}$:

$$\overline{\mathbb{N}} \triangleq \mathbb{N} \cup \{+\infty\} \qquad \mathcal{W} \triangleq \bigcup_{n \in \overline{\mathbb{N}}} \left([\![1, n]\!] \to \mathcal{R} \right)$$

Maximal Rewrite Words of $\mathcal{R}$, Originating in Π:

${}^{\mathcal{R}}(\!|\Pi|\!)$ is the set of words $w \in \mathcal{W}$ such that

$$\exists u_0 \in \Pi : \exists u_1, \ldots, u_{\#w} \in \mathcal{T}(\mathbb{A}) : \forall k \in \operatorname{dom} w,$$
$$u_{k-1} \xrightarrow{w(k)} u_k \ \wedge \ \#w \in \mathbb{N} \Rightarrow \mathcal{R}(\{u_{\#w}\}) = \varnothing$$

Notations:

Length $\#w \in \overline{\mathbb{N}}$ of a word w : $\#w \triangleq \operatorname{Card}(\operatorname{dom} w)$.

Formula $\varphi \in$ LTL: $\approx$ Finite-LTL [4]

$$\varphi := X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \bullet^m \varphi \mid \circ^m \varphi \mid \varphi \,\mathbf{U}\, \varphi \qquad X \in \wp(\mathcal{R})$$
$$\top \mid \bot \mid \varphi \vee \varphi \mid \varphi \Rightarrow \varphi \mid \Diamond\varphi \mid \Box\varphi \qquad m \in \mathbb{N}\,.$$

Semantics of LTL:

$$\begin{array}{lll}
(w,i) \models X & \text{iff} & i \in \operatorname{dom} w \text{ and } w(i) \in X \\
(w,i) \models \neg\varphi & \text{iff} & (w,i) \not\models \varphi \\
(w,i) \models (\varphi \wedge \psi) & \text{iff} & (w,i) \models \varphi \text{ and } (w,i) \models \psi \\
(w,i) \models \bullet^m \varphi & \text{iff} & i + m \in \operatorname{dom} w \text{ and } (w, i+m) \models \varphi \\
(w,i) \models \circ^m \varphi & \text{iff} & i + m \notin \operatorname{dom} w \text{ or } (w, i+m) \models \varphi \\
(w,i) \models \varphi \,\mathbf{U}\, \psi & \text{iff} & \left\{ \begin{array}{l} \exists j \in \operatorname{dom} w : j \geqslant i \wedge (w,j) \models \psi \\ \qquad \wedge\ \forall k \in [\![i, j-1]\!],\ (w,k) \models \varphi \end{array} \right.
\end{array}$$

For any $w \in \mathcal{W}$, $i \in \mathbb{N}_1$, $m \in \mathbb{N}$ and $X \in \wp(\mathcal{R})$.

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For any $w \in \mathcal{W}$, $i \in \mathbb{N}_1$, $m \in \mathbb{N}$ and $X \in \wp(\mathcal{R})$.

Satisfaction:

Let φ be an LTL formula:

- $w \models \varphi \iff (w,1) \models \varphi$
- $\mathcal{R}, \Pi \models \varphi \iff \forall w \in {}^{\mathcal{R}}(\!|\Pi|\!),\ w \models \varphi$

Rewrite Proposition on $\mathcal{R}$, from Π

$$X \in \wp(\mathcal{R})$$

$$\pi := \gamma \mid \gamma \wedge \gamma \mid \gamma \vee \gamma \qquad \gamma := \ell = \varnothing \mid X \subseteq X \mid \ell \subseteq \ell$$

$$\ell := \Pi \mid \mathcal{T}(\mathbb{A}) \mid X(\ell) \mid X^{-1}(\ell) \mid X^{*}(\ell)$$

A RP π has a trivial **truth value**.

Problem Statement

Input: $\mathcal{R}$, $\varphi \in \mathrm{LTL}$, $\Pi \subseteq \mathcal{T}(\mathbb{A})$

Output: RP π such that either

1. *exact translation*: $\mathcal{R}, \Pi \models \varphi \iff \pi$
2. *under-approximated translation*: $\mathcal{R}, \Pi \models \varphi \impliedby \pi$
3. *over-approximated translation*: $\mathcal{R}, \Pi \models \varphi \implies \pi$

Intuition: Weak & Strong, Past & Future

- $\mathcal{R}, \Pi \models \neg X$:

$$(w, i) \models \neg X \iff i \in \mathrm{dom}\, w \Rightarrow w(i) \notin X$$

$$\pi_1 \iff \mathcal{R}, \Pi \models \neg X \iff \forall w \in {}^{\mathcal{R}}(\!|\Pi|\!),\ (w, 1) \models \neg X$$

$$\pi_1 \equiv X(\Pi) = \varnothing$$

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$$(w, i) \models X \iff i \in \mathrm{dom}\, w \ \wedge \ w(i) \in X$$

$\pi_2 \equiv [\mathcal{R} \setminus X](\Pi) = \varnothing$ **?**

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- **$\Box\neg X$:**

$$(w, i) \models \Box\varphi \iff \forall j \in \operatorname{dom} w,\ j \geqslant i \Rightarrow (w, j) \models \varphi$$

$$(w, i) \models \neg X \iff i \notin \operatorname{dom} w \vee w(i) \notin X$$

$$\pi_3 \equiv X\left(\mathcal{R}^*(\Pi)\right) = \varnothing$$

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$\pi_3 \equiv X\left(\mathcal{R}^*(\Pi)\right) = \varnothing \equiv \pi_1[\mathcal{R}^*(\Pi)/\Pi]$

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- **$\Box X$:**

$$\begin{aligned}\pi_4 &\equiv \pi_2'[\mathcal{R}^*(\Pi)/\Pi] \\ &\equiv [\mathcal{R} \setminus X]\left(\mathcal{R}^*(\Pi)\right) = \varnothing \ \wedge \ \mathcal{R}^*(\Pi) \subseteq X^{-1}(\mathcal{T}(\mathbb{A}))\end{aligned}$$

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WRONG!

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$$\varphi : \pi \equiv \pi \iff \mathcal{R}, \Pi \models \varphi \iff \forall w \in {}^{\mathcal{R}}(\!|\Pi|\!),\ w \models \varphi$$

$$\forall w \in {}^{\mathcal{R}}(\!|\Pi|\!),\ w \models \varphi \ \wedge \ \forall w \in {}^{\mathcal{R}}(\!|\Pi|\!),\ w \models \psi$$

$$\iff$$

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- **Disjunction:** $\pi_5 \vee \pi_5' \implies \mathcal{R}, \Pi \models \varphi \vee \psi$
- **Negation:** $\mathcal{R}, \Pi \not\models \varphi \neq \mathcal{R}, \Pi \models \neg\varphi$: NNF required

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  $X : \pi_2', Y : \pi_2'' \equiv \pi_2'[Y/X], \pi_7 \equiv \pi_2''[X(\Pi)/\Pi]$
  $\Box(X \Rightarrow \bullet Y) : \pi_0 \equiv \pi_7[\mathcal{R}^*(X(\Pi))/X(\Pi)]$

Intuition: Weak & Strong, Past & Future

- **¬X:** $\pi_1 \equiv X(\Pi) = \varnothing$
- **X:** $\pi_2{}' \equiv [\mathcal{R} \setminus X](\Pi) = \varnothing \wedge \Pi \subseteq X^{-1}(\mathcal{T}(\mathbb{A}))$
- **□¬X:** $\pi_3 \equiv X\left(\mathcal{R}^*(\Pi)\right) = \varnothing \equiv \pi_1[\mathcal{R}^*(\Pi)/\Pi]$
- **□X:** $\pi_4' \equiv [\mathcal{R} \setminus X]\left(\mathcal{R}^*(\Pi)\right) = \varnothing$
- **Conjunction:** if $\varphi : \pi_5$ and $\psi : \pi_5'$ then $\varphi \wedge \psi : \pi_5 \wedge \pi_5'$.
- **Disjunction:** $\pi_5 \vee \pi_5' \implies \mathcal{R}, \Pi \models \varphi \vee \psi$
- **Negation:** $\mathcal{R}, \Pi \not\models \varphi \neq \mathcal{R}, \Pi \models \neg\varphi$: NNF required
- **Implication:** $X \Rightarrow \bullet Y$:
  $\pi_7 \equiv [\mathcal{R} \setminus Y]\left(X(\Pi)\right) = \varnothing \wedge X(\Pi) \subseteq Y^{-1}\left(\mathcal{T}(\mathbb{A})\right)$
  $X : \pi_2', Y : \pi_2'' \equiv \pi_2'[Y/X], \pi_7 \equiv \pi_2''[X(\Pi)/\Pi]$
  $\square(X \Rightarrow \bullet Y) : \pi_0 \equiv \pi_7[\mathcal{R}^*(X(\Pi))/X(\Pi)]$
  What about $\bullet Y \Rightarrow X$?

Restricting the Fragment

Not Everything Can Be Translated

$\mathcal{R}^*(\Pi)$ hides traces: $\Diamond X$ probably untranslatable.
So is "Until" family: $\{\Diamond, \mathbf{U}, \mathbf{W}, \mathbf{R}, \ldots\}$.

Restricted Fragment: $\mathcal{R}$-LTL

$$\varphi := X \mid \neg X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \Rightarrow \varphi \mid \qquad X \in \wp(\mathcal{R})$$
$$\bullet^m \varphi \mid \circ^m \varphi \mid \Box \varphi \qquad m \in \mathbb{N} .$$

Restricting the Fragment
Not Everything Can Be Translated

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Restricted Fragment: $\mathcal{R}$-LTL

$$\varphi := X \mid \neg X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \Rightarrow \varphi \mid \bullet^m \varphi \mid \circ^m \varphi \mid \Box \varphi \qquad X \in \wp(\mathcal{R}),\ m \in \mathbb{N}.$$

$\varphi \implies \psi$: working restriction of φ to

Restricted Antecedent Fragment: $\mathcal{A}$-LTL

$$\varphi := X \mid \neg X \mid \varphi \wedge \varphi \mid \bullet^m \varphi \mid \circ^m \varphi \mid \Box \varphi \qquad X \in \wp(\mathcal{R}),\ m \in \mathbb{N}.$$

$\vee$ handled outside signatures, left-assoc $\Rightarrow$ chains not handled

Weak & Strong Semantics

Bookkeeping For The Context

$$
\begin{array}{lll}
(w,i) \models^{s} X & \text{iff} & i \in \operatorname{dom} w \text{ and } w(i) \in X \\
(w,i) \models^{w} X & \text{iff} & i \notin \operatorname{dom} w \text{ or } w(i) \in X \\
(w,i) \models^{\mu} \neg X & \text{iff} & i \notin \operatorname{dom} w \text{ or } w(i) \notin X \\
(w,i) \models^{\mu} (\varphi \vee \psi) & \text{iff} & (w,i) \models^{\mu} \varphi \text{ or } (w,i) \models^{\mu} \psi \\
(w,i) \models^{\mu} (\varphi \wedge \psi) & \text{iff} & (w,i) \models^{\mu} \varphi \text{ and } (w,i) \models^{\mu} \psi \\
(w,i) \models^{\mu} (\varphi \Rightarrow \psi) & \text{iff} & (w,i) \models^{s} \varphi \implies (w,i) \models^{s} \psi \\
(w,i) \models^{\mu} \bullet^{m} \varphi & \text{iff} & i+m \in \operatorname{dom} w \text{ and } (w,i+m) \models^{s} \varphi \\
(w,i) \models^{\mu} \circ^{m} \varphi & \text{iff} & i+m \notin \operatorname{dom} w \text{ or } (w,i+m) \models^{w} \varphi \\
(w,i) \models^{\mu} \Box \varphi & \text{iff} & \forall j \in \operatorname{dom} w,\ j \geqslant i \Rightarrow (w,j) \models^{w} \varphi
\end{array}
$$

For any $m \in \mathbb{N}$, $\mu \in \{w, s\}$

$$i \in \operatorname{dom} w \implies (w,i) \models^{s} \varphi \Leftrightarrow (w,i) \models^{w} \varphi$$

$$(w,i) \models^{s} \varphi \iff (w,i) \models \varphi$$

Signatures

Implication: Girdling the Future

Idea: $\varphi \Rightarrow \psi$? φ as an *assumption*, i.e. a *model* of φ: $\xi(\varphi)$

$$\Sigma = \bigcup_{n \in \mathbb{N}} \Big[\big([\![1, n]\!] \cup \{\omega\} \big) \to \wp(\mathcal{R}) \Big] \times \wp(\overline{\mathbb{N}}) \ .$$

Notations:

- compactly as $\sigma = \wr f \mid S \int = \wr \partial\sigma \mid \nabla\sigma \int$,
- or *in extenso* as $\wr f(1), f(2), \ldots, f(\#\sigma) \fatsemi f(\omega) \mid S \int$.

Example: $\xi\big(X \wedge \circ^1 Y \wedge \circ^2 \Box Z\big) = \wr X, Y \fatsemi Z \mid \overline{\mathbb{N}}_1 \int$

Signatures

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- compactly as $\sigma = \wr f \mid S \int = \wr \partial\sigma \mid \nabla\sigma \int$,
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Example: $\xi\big(X \wedge \circ^1 Y \wedge \circ^2 \Box Z\big) = \wr X, Y \fatsemi Z \mid \overline{\mathbb{N}}_1 \int$

Constrained Words:

$${}^{\mathcal{R}}(\!|\Pi \fatsemi \sigma|\!) \triangleq \big\{ w \in {}^{\mathcal{R}}(\!|\Pi|\!) \ \big| \ \#w \in \nabla\sigma \ \wedge \ \forall k \in \operatorname{dom} w, \ w(k) \in \sigma[k] \big\}$$

$$\forall\ \Pi \subseteq \mathcal{T}(\mathbb{A}), \varphi \in \mathcal{A}\text{-LTL}, \ {}^{\mathcal{R}}(\!|\Pi \fatsemi \xi(\varphi)|\!) = \big\{ w \in {}^{\mathcal{R}}(\!|\Pi|\!) \ \big| \ w \models \varphi \big\}$$

Signatures

Implication: Girdling the Future

$$\xi(\top) \triangleq \wr ; \mathcal{R} \mid \overline{\mathbb{N}} \int = \varepsilon \qquad \xi(\bot) \triangleq \wr ; \varnothing \mid \varnothing \int$$

$$\xi(X) \triangleq \wr X ; \mathcal{R} \mid \overline{\mathbb{N}}_1 \int \qquad \xi(\neg X) \triangleq \wr \mathcal{R} \setminus X ; \mathcal{R} \mid \overline{\mathbb{N}} \int$$

$$\xi(\bullet^m \varphi) \triangleq \xi(\varphi) \blacktriangleright m \qquad \xi(\circ^m \varphi) \triangleq \xi(\varphi) \rhd m$$

$$\xi(\varphi \wedge \psi) \triangleq \xi(\varphi) \otimes \xi(\psi) \qquad \xi(\Box \varphi) \triangleq \bigotimes_{m=0}^{\infty} \Big[\xi(\varphi) \rhd m \Big]$$

- $\sigma \blacktriangleright m =$ **Strong Shift Right** $=$
  $\wr \mathcal{R}_1, \ldots, \mathcal{R}_m, \partial\sigma(1), \ldots, \partial\sigma(\#\sigma) ; \partial\sigma(\omega) \mid (\nabla\sigma \setminus \{0\}) + m \int$
- $\sigma \rhd m =$ **Weak Shift Right** $=$
  $\wr \mathcal{R}_1, \ldots, \mathcal{R}_m, \partial\sigma(1), \ldots, \partial\sigma(\#\sigma) ; \partial\sigma(\omega) \mid [\![0, m]\!] \cup (\nabla\sigma + m) \int$

Signatures
Implication: Girdling the Future

$$\xi(\top) \triangleq \wr ; \mathcal{R} \mid \overline{\mathbb{N}} \int = \varepsilon \qquad \xi(\bot) \triangleq \wr ; \varnothing \mid \varnothing \int$$

$$\xi(X) \triangleq \wr X ; \mathcal{R} \mid \overline{\mathbb{N}}_1 \int \qquad \xi(\neg X) \triangleq \wr \mathcal{R} \setminus X ; \mathcal{R} \mid \overline{\mathbb{N}} \int$$

$$\xi(\bullet^m \varphi) \triangleq \xi(\varphi) \blacktriangleright m \qquad \xi(\circ^m \varphi) \triangleq \xi(\varphi) \rhd m$$

$$\xi(\varphi \wedge \psi) \triangleq \xi(\varphi) \otimes \xi(\psi) \qquad \xi(\Box \varphi) \triangleq \bigotimes_{m=0}^{\infty} \Big[\xi(\varphi) \rhd m \Big]$$

- $\sigma \blacktriangleright m$ = **Strong Shift Right** =
  $\wr \mathcal{R}_1, \ldots, \mathcal{R}_m, \partial\sigma(1), \ldots, \partial\sigma(\#\sigma) ; \partial\sigma(\omega) \mid (\nabla\sigma \setminus \{0\}) + m \int$
- $\sigma \rhd m$ = **Weak Shift Right** =
  $\wr \mathcal{R}_1, \ldots, \mathcal{R}_m, \partial\sigma(1), \ldots, \partial\sigma(\#\sigma) ; \partial\sigma(\omega) \mid [\![0, m]\!] \cup (\nabla\sigma + m) \int$

Signatures: Product

Definition: Signature Product

$\sigma \otimes \sigma' \triangleq \wr g \mid \nabla\sigma \cap \nabla\sigma' \int$, where

$$g \triangleq \left| \begin{array}{ccc} \operatorname{dom} \partial\sigma \cup \operatorname{dom} \partial\sigma' & \longrightarrow & \wp(\mathcal{R}) \\ k & \longmapsto & \sigma[k] \cap \sigma'[k] \end{array} \right. .$$

Consequence: $\forall k \in \mathbb{N}_1,\ (\sigma \otimes \sigma')[k] = \sigma[k] \cap \sigma'[k]$

Theorem: ${}^{\mathcal{R}}(\!|\Pi \fatsemi \sigma \otimes \sigma'|\!) = {}^{\mathcal{R}}(\!|\Pi \fatsemi \sigma|\!) \cap {}^{\mathcal{R}}(\!|\Pi \fatsemi \sigma'|\!)$

Example: $\sigma = \wr X, Y \fatsemi Z \mid \mathbb{N}_2 \int \qquad \rho = \wr X' \fatsemi Z' \mid \mathbb{N}_3 \int$

$$\sigma \otimes \rho = \wr X \cap X', Y \cap Z' \fatsemi Z \cap Z' \mid \mathbb{N}_3 \int$$

Signatures: Convergence

$\rho = (\sigma_n)_{n\in\mathbb{N}}$ **converges** if

1. $\nabla\sigma_n \to_n \nabla\sigma_\infty$
2. for all $k \in \mathbb{N}_1$, $\sigma_n[k] \to_n \sigma_\infty[k]$
3. $\sigma_\infty[k] \to_{k\geqslant 1} \sigma_\infty[\infty]$

$$\sigma_\infty \triangleq \lim_{n\to\infty} \sigma_n \triangleq \wr \sigma_\infty[1], \ldots, \sigma_\infty[N] \S \sigma_\infty[\infty] \mid \nabla\sigma_\infty \int$$

Example:
$(\wr \mathcal{R}_1, \ldots \mathcal{R}_n, X \S \mathcal{R} \mid [\![1, n]\!] \int)_{n\in\mathbb{N}}$, with $\mathcal{R}_i = \mathcal{R}\ \forall i$, converges towards $\wr \S X \mid \mathbb{N} \int$.

Signatures: Infinite Products

Remark: $(\Sigma, \otimes, \varepsilon)$ is a commutative monoid.
Notation: $\bigotimes_{k=l}^{m} \sigma_k \triangleq \sigma_l \otimes \sigma_{l+1} \otimes \cdots \otimes \sigma_m$
Definition: $\bigotimes_{k=l}^{\infty} \sigma_k$ converges $\iff$ $(\bigotimes_{k=l}^{n} \sigma_k)_{n \in \mathbb{N}_l}$ converges

$$\bigotimes_{k=l}^{\infty} \sigma_k \triangleq \lim_{n \to \infty} \bigotimes_{k=l}^{n} \sigma_k \ .$$

Lemmas: Breaking Infinite Products, Automatic Convergence

$$\mathcal{R}(\!|\Pi \fatsemi \bigotimes_{n=0}^{\infty} \sigma_n|\!) = \bigcap_{n=0}^{\infty} \mathcal{R}(\!|\Pi \fatsemi \sigma_n|\!)$$

$$\bigotimes_{n=0}^{\infty} [\sigma \blacktriangleright n] \ , \quad \bigotimes_{n=0}^{\infty} [\sigma \rhd n] \quad \text{conv. } \forall \sigma$$

Translation Blocks and Rules

Block: $\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \iff \forall w \in {}^{\mathcal{R}}(\!|\Pi \fatsemi \sigma|\!),\ w \models^{\mu} \varphi$

Theorem: $\langle \Pi \fatsemi \varepsilon \ \Vdash^{s} \ \varphi \rangle \iff \mathcal{R}, \Pi \models \varphi$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \qquad P(\sigma, \varphi)}{\pi} \quad \text{or} \quad \uparrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \qquad P(\sigma, \varphi)}{\pi}$$

$\Upsilon \in$ translation blocks $\qquad P \in \Sigma \times \mathcal{R}\text{-LTL} \to \mathbb{B}$

$$\pi := \gamma \mid \gamma \wedge \gamma \mid \gamma \vee \gamma \qquad \gamma := \ell = \varnothing \mid X \subseteq X \mid \ell \subseteq \ell \mid \Upsilon$$

$$\ell := \Pi \mid \mathcal{T}(\mathbb{A}) \mid X(\ell) \mid X^{-1}(\ell) \mid X^{*}(\ell)$$

Semantics:

- $\updownarrow$**-rules:** $P(\sigma, \varphi) \implies \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \Leftrightarrow \pi$
- $\uparrow$**-rules:** $P(\sigma, \varphi) \implies \pi \Rightarrow \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \top \rangle}{\top} \quad (\top) \qquad \updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bot \rangle}{\bot} \quad (\bot)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \wedge Y \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \cap Y \rangle} \quad (\wedge x) \qquad \updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \vee Y \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \cup Y \rangle} \quad (\vee x)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \wedge \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \wedge \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \psi \rangle} \quad (\wedge)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ [\varphi \vee \varphi'] \Rightarrow \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \Rightarrow \psi \rangle \wedge \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi' \Rightarrow \psi \rangle} \quad (\vee^{\Rightarrow}_{\wedge})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \vee \psi \rangle \qquad \neg\varphi \in \mathcal{A}\text{-LTL}}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \neg\varphi \Rightarrow \psi \rangle} \quad (\vee^{\neg}_{\Rightarrow})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \top \rangle}{\top} \quad (\top) \qquad \updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bot \rangle}{\bot} \quad (\bot)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \wedge Y \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \cap Y \rangle} \quad (\wedge x) \qquad \updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \vee Y \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \cup Y \rangle} \quad (\vee x)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \wedge \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \wedge \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \psi \rangle} \quad (\wedge)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ [\varphi \vee \varphi'] \Rightarrow \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \Rightarrow \psi \rangle \wedge \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi' \Rightarrow \psi \rangle} \quad (\vee^{\Rightarrow}_{\wedge})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \vee \psi \rangle \qquad \neg\varphi \in \mathcal{A}\text{-LTL}}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \neg\varphi \Rightarrow \psi \rangle} \quad (\vee^{\neg}_{\Rightarrow})$$

$$\uparrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \vee \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \vee \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \psi \rangle} \quad (\vee_{\uparrow})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \Rightarrow \psi \rangle}{\langle \Pi \fatsemi \sigma \otimes \xi(\varphi) \ \Vdash^{s} \ \psi \rangle} \quad (\Rightarrow_{\Sigma})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \circ^{m} \varphi \rangle}{\langle \Pi^{m}_{\sigma} \fatsemi \sigma \blacktriangleleft m \ \Vdash^{w} \ \varphi \rangle} \quad (\circ^{m})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bullet^{m} \varphi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \circ^{m} \varphi \rangle \wedge \bigwedge_{n \in [\![0,m]\!] \cap \nabla\sigma} \Psi^{\sigma}_{\Pi}(n)} \quad (\bullet^{m})$$

$$\uparrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \vee \psi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \rangle \vee \langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \psi \rangle} \quad (\vee_{\uparrow})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \varphi \Rightarrow \psi \rangle}{\langle \Pi \fatsemi \sigma \otimes \xi(\varphi) \ \Vdash^{\mathbf{s}} \ \psi \rangle} \quad (\Rightarrow_{\Sigma})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \circ^{m} \varphi \rangle}{\langle \Pi^{m}_{\sigma} \fatsemi \sigma \blacktriangleleft m \ \Vdash^{w} \ \varphi \rangle} \quad (\circ^{m})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bullet^{m} \varphi \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \circ^{m} \varphi \rangle \wedge \bigwedge_{n \in [\![0,m]\!] \cap \nabla\sigma} \Psi^{\sigma}_{\Pi}(n)} \quad (\bullet^{m})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \Box\, \varphi \rangle \qquad \sigma \text{ is stable}}{\langle \sigma[\omega]^{*}(\Pi) \fatsemi \star\sigma \ \Vdash^{w} \ \varphi \rangle} \qquad (\Box_{*})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \Box\, \varphi \rangle \qquad \hbar\sigma \in \mathbb{N}_1}{\left\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bigwedge_{k=0}^{\hbar\sigma-1} \circ^{k} \varphi \right\rangle \wedge \left\langle \Pi_{\sigma}^{\hbar\sigma} \fatsemi \sigma \lhd \hbar\sigma \ \Vdash^{\mu} \ \Box\, \varphi \right\rangle} \qquad (\Box_{\hbar})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \varepsilon \ \Vdash^{\mu} \ \Box\, \varphi \rangle}{\langle \mathcal{R}^{*}(\Pi) \fatsemi \star\varepsilon \ \Vdash^{w} \ \varphi \rangle} \qquad (\text{e.g. })$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \neg X \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{w} \ \mathcal{R} \setminus X \rangle} \qquad (\neg X)$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \Box\, \varphi \rangle \qquad \sigma \text{ is stable}}{\langle \sigma[\omega]^{*}(\Pi) \fatsemi \star\sigma \ \Vdash^{w} \ \varphi \rangle} \qquad (\Box_{*})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \Box\, \varphi \rangle \qquad \hbar\sigma \in \mathbb{N}_1}{\left\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \bigwedge_{k=0}^{\hbar\sigma-1} \circ^{k} \varphi \right\rangle \wedge \left\langle \Pi^{\hbar\sigma}_{\sigma} \fatsemi \sigma \lhd \hbar\sigma \ \Vdash^{\mu} \ \Box\, \varphi \right\rangle} \qquad (\Box_{\hbar})$$

$$\updownarrow \frac{\langle \Pi \fatsemi \varepsilon \ \Vdash^{\mu} \ \Box\, \varphi \rangle}{\langle \mathcal{R}^{*}(\Pi) \fatsemi \star\varepsilon \ \Vdash^{w} \ \varphi \rangle} \qquad (\text{e.g. })$$

$$\updownarrow \frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ \neg X \rangle}{\langle \Pi \fatsemi \sigma \ \Vdash^{w} \ \mathcal{R} \setminus X \rangle} \qquad (\neg X)$$

Hybrid Rules
(Work In Progress)

$$?\frac{\langle \Pi \fatsemi \sigma \ \Vdash^{w} \ X \rangle \qquad \uparrow \ell\sigma \leqslant 1 \ \updownarrow \sigma \lhd 1 = \varepsilon}{\left[\mathcal{R} \setminus (X \cap \sigma[1])\right](\Pi) = \varnothing} \qquad (X^{w}_{\ell \leqslant 1})$$

$$?\frac{\langle \Pi \fatsemi \sigma \ \Vdash^{s} \ X \rangle \qquad \uparrow \ell\sigma = 0 \ \updownarrow \sigma \lhd 1 = \varepsilon}{\langle \Pi \fatsemi \sigma \ \Vdash^{w} \ X \rangle \wedge \Pi \subseteq \left(X \cap \sigma[1]\right)^{-1}(\mathcal{T}(\mathbb{A}))} \qquad (X^{s}_{\ell 0})$$

$$?\frac{\langle \Pi \fatsemi \sigma \ \Vdash^{s} \ X \rangle \qquad \uparrow \ell\sigma = 1 \ \updownarrow \sigma \lhd 1 = \varepsilon}{\langle \Pi \fatsemi \sigma \ \Vdash^{w} \ X \rangle} \qquad (X^{s}_{\ell 1})$$

$$?\frac{\langle \Pi \fatsemi \sigma \ \Vdash^{\mu} \ X \rangle \qquad \uparrow \ell\sigma \geqslant 2 \ \updownarrow \sigma \lhd \ell\sigma = \varepsilon}{\sigma[\ell\sigma]\left(\cdots \sigma[2]\left(\left[\mathcal{R} \setminus (X \cap \sigma[1])\right](\Pi)\right)\cdots\right) = \varnothing} \qquad (X^{\mu}_{\ell 2})$$

Hybrid Rules
(Work In Progress)

$$?\frac{\langle \Pi \fatsemi \sigma \Vdash^{w} X \rangle \qquad \uparrow \ell\sigma \leqslant 1 \updownarrow \sigma \lhd 1 = \varepsilon}{\left[\mathcal{R} \setminus (X \cap \sigma[1])\right](\Pi) = \varnothing} \qquad (X^{w}_{\ell\leqslant 1})$$

$$?\frac{\langle \Pi \fatsemi \sigma \Vdash^{s} X \rangle \qquad \uparrow \ell\sigma = 0 \updownarrow \sigma \lhd 1 = \varepsilon}{\langle \Pi \fatsemi \sigma \Vdash^{w} X \rangle \wedge \Pi \subseteq \left(X \cap \sigma[1]\right)^{-1}(\mathcal{T}(\mathbb{A}))} \qquad (X^{s}_{\ell 0})$$

$$?\frac{\langle \Pi \fatsemi \sigma \Vdash^{s} X \rangle \qquad \uparrow \ell\sigma = 1 \updownarrow \sigma \lhd 1 = \varepsilon}{\langle \Pi \fatsemi \sigma \Vdash^{w} X \rangle} \qquad (X^{s}_{\ell 1})$$

$$?\frac{\langle \Pi \fatsemi \sigma \Vdash^{\mu} X \rangle \qquad \uparrow \ell\sigma \geqslant 2 \updownarrow \sigma \lhd \ell\sigma = \varepsilon}{\sigma[\ell\sigma]\left(\cdots \sigma[2]\left(\left[\mathcal{R} \setminus (X \cap \sigma[1])\right](\Pi)\right)\cdots\right) = \varnothing} \qquad (X^{\mu}_{\ell 2})$$

Example: Derivation

$$\updownarrow \frac{\langle \Pi \fatsemi \varepsilon \Vdash^{s} \Box(X \Rightarrow \bullet^{1} Y)\rangle}{\updownarrow \frac{\langle \mathcal{R}^{*}(\Pi) \fatsemi \star\varepsilon \Vdash^{w} X \Rightarrow \bullet^{1} Y\rangle}{\updownarrow \frac{\langle \mathcal{R}^{*}(\Pi) \fatsemi (\!| X \fatsemi \mathcal{R} \mid \overline{\mathbb{N}}_1 |\!) \Vdash^{s} \bullet^{1} Y\rangle}{\Psi^{(\!| X \fatsemi \mathcal{R} \mid \overline{\mathbb{N}}_1 |\!)}_{\mathcal{R}^{*}(\Pi)}(1) \wedge \updownarrow \frac{\langle \mathcal{R}^{*}(\Pi) \fatsemi (\!| X \fatsemi \mathcal{R} \mid \overline{\mathbb{N}}_1 |\!) \Vdash^{s} \circ^{1} Y\rangle}{\updownarrow \frac{\langle X(\mathcal{R}^{*}(\Pi)) \fatsemi (\!| \fatsemi \mathcal{R} \mid \overline{\mathbb{N}}_1 |\!) \Vdash^{w} Y\rangle\,(X^{w}_{\ell \leqslant 1})}{[\mathcal{R} \setminus Y]\,(X(\mathcal{R}^{*}(\Pi))) = \varnothing}}{(\circ^{m})}}{(\bullet^{m})}}{(\Rightarrow_{\Sigma})}}{(\Box_{*})}$$

Yields:

$$[\mathcal{R} \setminus Y]\,(X(\mathcal{R}^{*}(\Pi))) = \varnothing \wedge \Psi^{(\!| X \fatsemi \mathcal{R} \mid \overline{\mathbb{N}}_1 |\!)}_{\mathcal{R}^{*}(\Pi)}(1)$$

$$[\mathcal{R} \setminus Y]\,(X(\mathcal{R}^{*}(\Pi))) = \varnothing \wedge X(\mathcal{R}^{*}(\Pi)) \subseteq \mathcal{R}^{-1}\,(\mathcal{T}(\mathbb{A}))$$

$$[\mathcal{R} \setminus Y]\,(X(\mathcal{R}^{*}(\Pi))) = \varnothing \wedge X(\mathcal{R}^{*}(\Pi)) \subseteq Y^{-1}\,(\mathcal{T}(\mathbb{A})),$$

Conclusion

Current Results:

1. Exact automatic translation on a fragment of LTL
2. (loose) Under-Approx on a slightly larger fragment

Next Steps:

1. Simplification: Get rid of weak/strong twin semantics (✓)
2. Refine base case "hybrid rules" (✓)
3. Generalise RP $\rightarrow$ semi-decision translation (✓)
4. Characterise translatable fragment of LTL
5. Generalise process to a larger fragment (in CTL*)


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