Loops and Overloops for Tree Walking Automata

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July 13, 2011

Tree Walking Automata

- Old formalism (≈1970, Aho & Ullman)
- Sequential model, as opposed to *branching* tree automata
- Less extensively studied model for a long while...
- ...but long standing questions solved in recent years
- Recent surge in interest, due mostly to connection to XML (Fragments of Core XPath, streaming etc)

- Research focused on fundamental problems (expressiveness...)
- *Our focus: practical, efficient algorithms*
- *Finality: efficient XML queries; compact Model-Checking*

Starting Point:

Transformation from TWA to BUTA

A **Tree-Walking Automaton** is a tuple $\mathcal{A} = \langle \Sigma, Q, I, F, \Delta \rangle$

$$\Delta \subseteq \Sigma \times Q \times \underbrace{\{\star, \boldsymbol{0}, \boldsymbol{1}\}}_{\mathbb{T}\ :\ \text{types}} \times \underbrace{\{\uparrow, \circlearrowleft, \swarrow, \searrow\}}_{\mathbb{M}\ :\ \text{moves}} \times Q$$

Notations

- "$\langle f, p, \tau \to \mu, q \rangle$" for the tuple $(f, p, \tau, \mu, q) \in \Delta$.
- $\langle \Sigma_2, p, \mathbb{T} \to \circlearrowleft, q \rangle \stackrel{\text{def}}{=} \{ (\sigma, p, \tau, \circlearrowleft, q) \mid \sigma \in \Sigma_2, \tau \in \mathbb{T} \}$

Remarks

- Ranked (binary) vs. unranked alphabet
- $\langle \Sigma_0, Q, \mathbb{T} \to \{\swarrow, \searrow\}, Q \rangle \cup \langle \Sigma, Q, \star \to \uparrow, Q \rangle$ invalid

A Very Simple TWA: $\mathcal{X} = \langle \Sigma, Q, I, F, \Delta \rangle$

- $\Sigma_0 = \{ a, b, c \}$ and $\Sigma_2 = \{ f, g, h \}$
- $Q = \{ q_\ell, q_u \}$, $I = \{q_\ell\}$, $F = \{q_u\}$

$$\Delta = \langle a, q_\ell, \{ \star, \mathbf{0} \} \to \circlearrowleft, q_u \rangle$$
$$\cup \langle \Sigma, q_u, \mathbf{0} \to \uparrow, q_u \rangle$$
$$\cup \langle \Sigma_2, q_\ell, \{ \star, \mathbf{0} \} \to \swarrow, q_\ell \rangle$$

$\mathcal{X}$ accepts exactly all trees whose left-most leaf is labelled by a — and the tree a itself.

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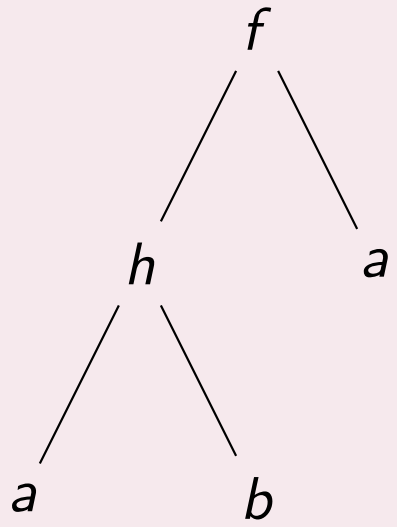

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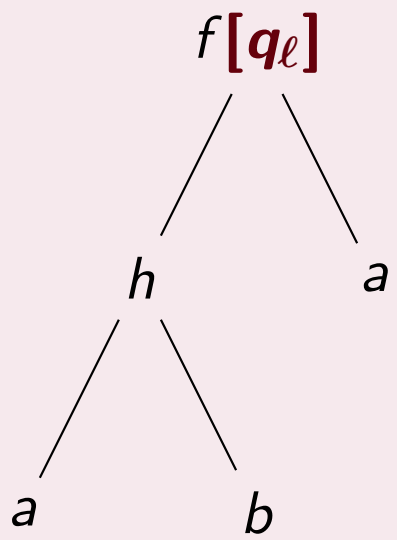

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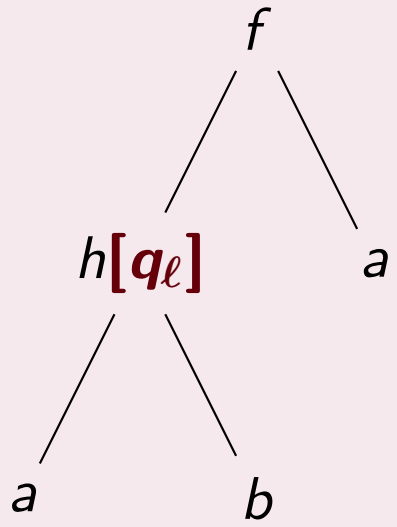

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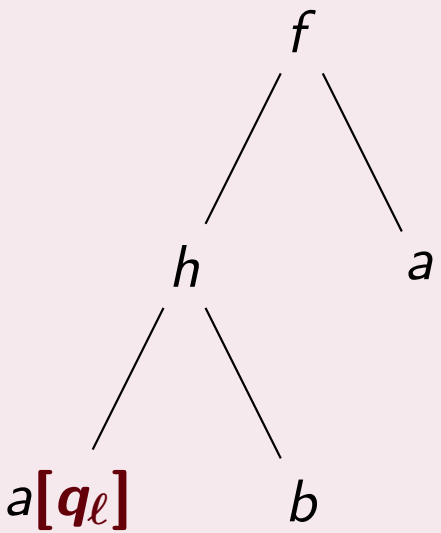

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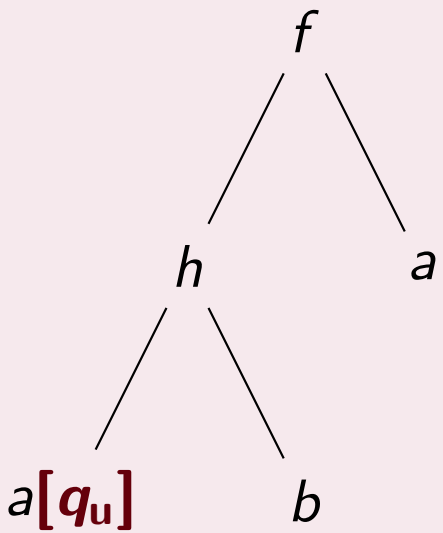

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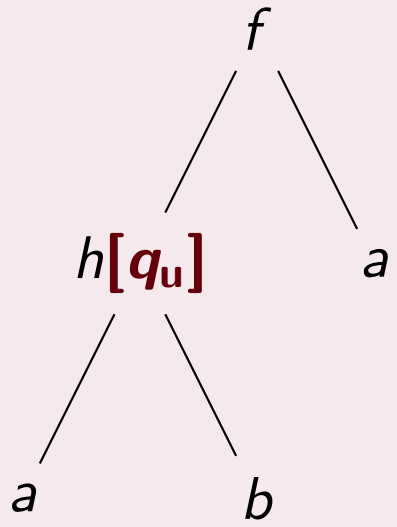

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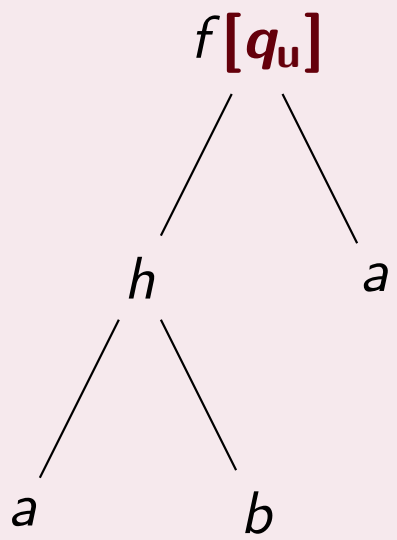


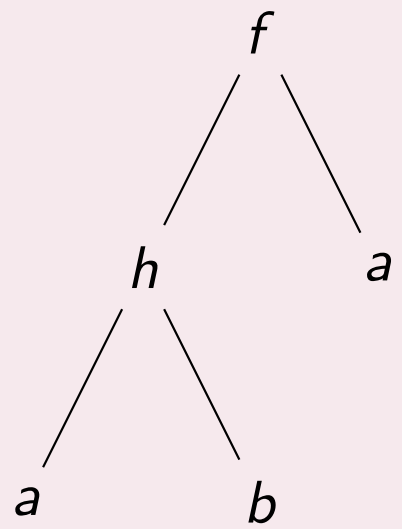
Given a TWA $\mathcal{A}$, build an equivalent BUTA $\mathcal{B}$.

- Known solution outlined in the literature [Bojańczyk, Samuelides]
- Based on the idea of **tree loops**
- Resulting states for $\mathcal{B}$: $\mathbb{T} \times 2^{Q^2}$ (or det. $(2^{Q^2})^{\mathbb{T}}$)

- Only proof sketches. No explicit algorithm is given.
- We argue that things are slightly less straightforward:
  - Needed states space: $\Sigma \times \mathbb{T} \times 2^{Q^2}$ (or det. $\Sigma \times (2^{Q^2})^{\mathbb{T}}$)
  - Because of this, some existing implementations are only *almost* correct [dtwa-tools]
- We introduce **tree overloops**
  - This time we really have $\mathbb{T} \times 2^{Q^2}$ (or det. $(2^{Q^2})^{\mathbb{T}}$)
  - Lower upper bound if $\mathcal{A}$ is deterministic: $|\mathbb{T}| \cdot 2^{|Q| \log_2(|Q|+1)}$

The Idea of Tree Loops

$(p, q) \in Q^2$ is a **loop** of $\mathcal{A}$ on $t|_\alpha$ if there exists a run which starts in p, ends in q (at the root α), and always stays in the subtree.



Loops of $\mathcal{X}$ on...

- t: $\{\}$
- $t|_0$: $\{\}$
- $t|_{0.0}$: $\{\}$
- $t|_{0.1}$: $\{\}$
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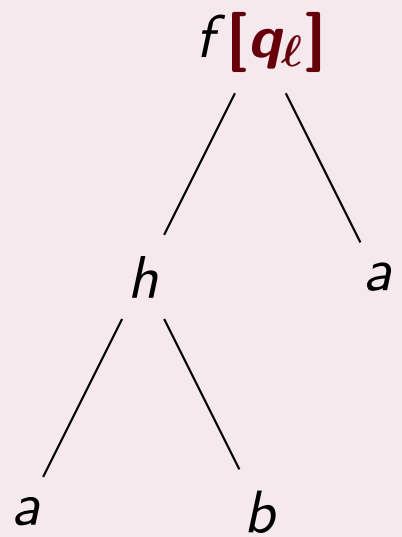

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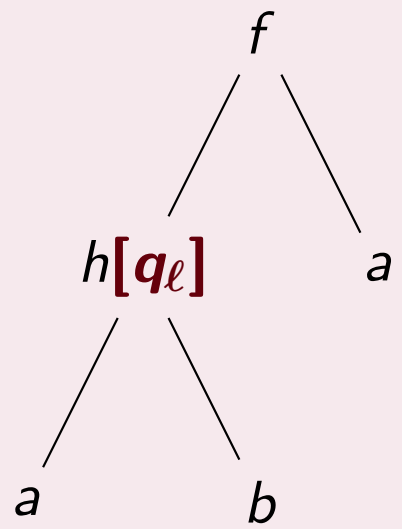

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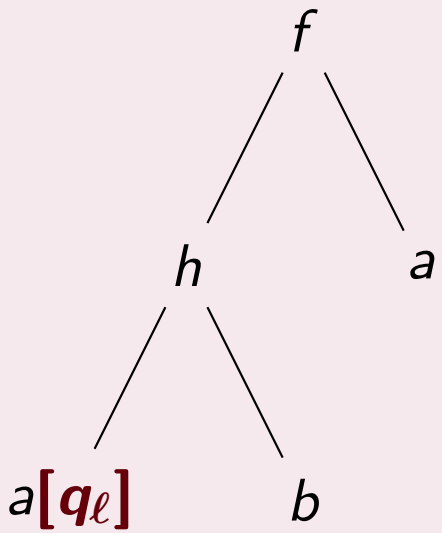

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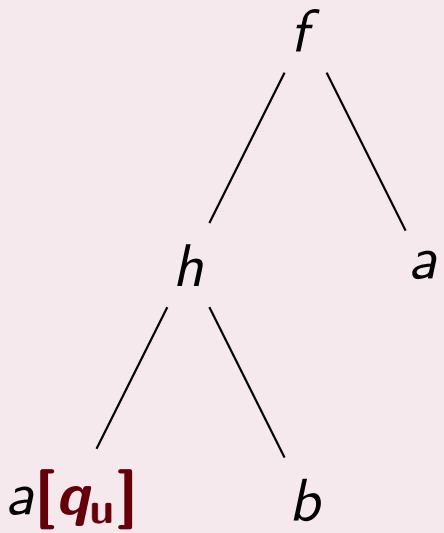

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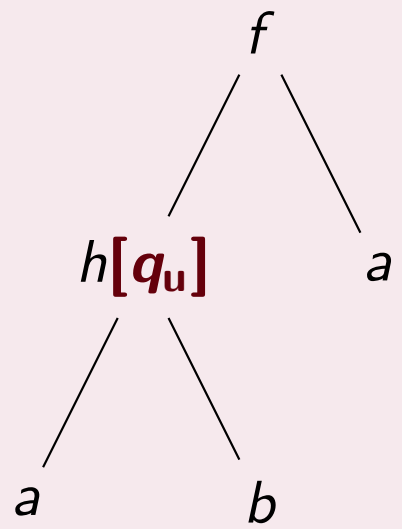

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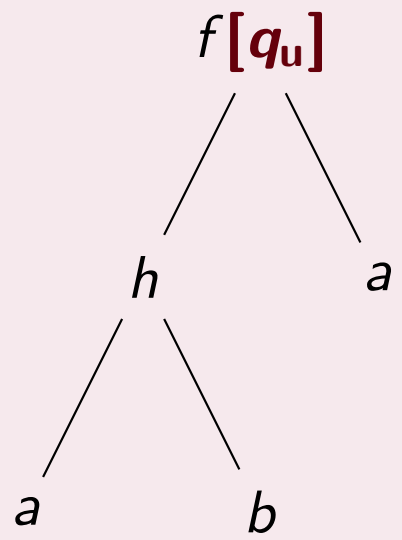

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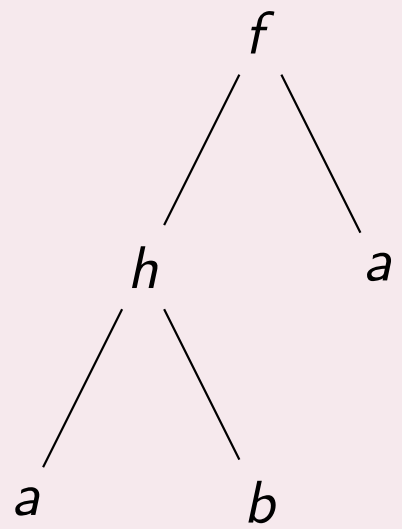

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A loop is a **simple loop** on $t|_\alpha$ if there is a run which forms it and reaches α exactly twice (*simple looping run*).

Proposition: Loops Decomposition

If $S \subseteq Q^2$ is the set of all simple loops of $\mathcal{A}$ on a given subtree $u = t|_\alpha$, then S^* is the set of all loops of $\mathcal{A}$ on u.

We denote $\mathcal{U}^\tau(u)$ the set of all loops of $\mathcal{A}$ on a subtree u, where τ is the type of the root of u. Compute loops on $u = a \in \Sigma_0$.

Simple looping run: run of the form $(\varepsilon, p) \twoheadrightarrow (\varepsilon, q)$ only.

$$\mathcal{H}^\tau_\sigma \stackrel{\text{def}}{=} \{\, (p,q) \mid \langle \sigma, p, \tau \to \circlearrowleft, q \rangle \in \Delta \,\}.$$

So we have $\mathcal{U}^\tau(a) = (\mathcal{H}^\tau_a)^*$.

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Let $f \in \Sigma_2$, and $u = f(u_0, u_1)$; root of type τ. Compute $\mathcal{U}^\tau(u)$.

First Move of a Simple Looping Run

- $\uparrow$ — impossible: leaves the subtree u
- $\circlearrowleft$ — all computed in $\mathcal{H}_f^\tau$
- $\swarrow$ — $(\varepsilon, p), (0, p_0), (\beta_1, s_1), \ldots, (\beta_n, s_n), (0, q_0), (\varepsilon, q)$, with all $\beta_k \trianglelefteq 0$. So $(p_0, q_0) \in \mathcal{U}^0(u_0)$.
- $\searrow$ —$(\varepsilon, p), (1, p_1), (\beta_1, s_1), \ldots, (\beta_n, s_n), (1, q_1), (\varepsilon, q)$, with all $\beta_k \trianglelefteq 1$. So $(p_1, q_1) \in \mathcal{U}^1(u_1)$.

Computing Tree Loops
Computation for Inner Nodes

Let $f \in \Sigma_2$, and $u = f(u_{\boldsymbol{0}}, u_{\boldsymbol{1}})$; root of type τ. Compute $\mho^\tau(u)$.

First Move of a Simple Looping Run

- $\swarrow$ — $(\varepsilon, p), (0, p_{\boldsymbol{0}}), (\beta_1, s_1), \ldots, (\beta_n, s_n), (0, q_{\boldsymbol{0}}), (\varepsilon, q)$, with all $\beta_k \trianglelefteq 0$. So $(p_{\boldsymbol{0}}, q_{\boldsymbol{0}}) \in \mho^{\boldsymbol{0}}(u_{\boldsymbol{0}})$.

To build a simple loop (p, q) on the subtree u, we need to...

1. choose a side: $\theta \in \mathbb{S} \stackrel{\text{def}}{=} \{\boldsymbol{0}, \boldsymbol{1}\}$
2. find an existing loop on that side: $(p_\theta, q_\theta) \in \mho^\theta(u_\theta)$
3. such that one can connect beginning and end
   1. $\langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta^a$ and
   2. $\langle u_\theta(\varepsilon), q_\theta, \theta \to \uparrow, q \rangle \in \Delta$

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[a] $\chi(\cdot) : \mathbb{S} \to \{\swarrow, \searrow\}$ such that $\chi(\boldsymbol{0}) = \swarrow$ and $\chi(\boldsymbol{1}) = \searrow$.

Loops on Leaves

Let $a \in \Sigma_0$, we have $\mho^\tau(a) = (\mathcal{H}_a^\tau)^*$.

Loops on Inner Nodes

Let $f \in \Sigma_2$, and $u = f(u_0, u_1)$; root of type τ. We have $\mho^\tau(u) =$

$$\left(\mathcal{H}_f^\tau \cup \left\{ (p,q) \;\middle|\; \begin{array}{c} \exists \theta \in \mathbb{S} : \\ \exists (p_\theta, q_\theta) \in \mho^\theta(u_\theta) \end{array} : \begin{array}{c} \langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta \\ \text{and} \\ \langle u_\theta(\varepsilon), q_\theta, \theta \to \uparrow, q \rangle \in \Delta \end{array} \right\} \right)^*$$

Transformation Into BUTA Using Loops

0. **Input:** A TWA $\mathcal{A} = \langle \Sigma, Q, I, F, \Delta \rangle$
1. **Initialise** *States* and *Rules* to $\varnothing$
2. **for each** $a \in \Sigma_0, \tau \in \mathbb{T}$ **do**
   - let $P = (a, \tau, \mathcal{H}_a^{\tau *})$
     add $a \to P$ to *Rules* and P to *States*
3. **repeat until** *Rules* remain unchanged
   - **for each** $f \in \Sigma_2, \tau \in \mathbb{T}$ **do**
     - add every $f(P_0, P_1) \to P$ to *Rules* and P to *States* where $P_0, P_1 \in States$ such that $P_0 = (\sigma_0, \mathbf{0}, S_0)$ and $P_1 = (\sigma_1, \mathbf{1}, S_1)$ and $P = (f, \tau, (\mathcal{H}_f^{\tau} \cup S)^*)$, with S the set of simple loops built on the sons.
4. **Output:** A BUTA $\mathcal{B}$ equivalent to $\mathcal{A}$:
   $\mathcal{B} = \langle \Sigma, States, \{ (\sigma, \star, L) \in States \mid L \cap (I \times F) \neq \varnothing \}, Rules \rangle$

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Using Loops

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Set of Simple Loops Built on the Sons

$$S = \left\{ (p,q) \;\middle|\; \begin{array}{c} \exists \theta \in \mathbb{S} : \\ \exists (p_\theta, q_\theta) \in S_\theta \end{array} \text{ st. } \begin{array}{c} \langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta \\ \text{and} \\ \langle \sigma_\theta, q_\theta, \theta \to \uparrow, q \rangle \in \Delta \end{array} \right\}$$

The son's symbol is needed to close the end of the loop.

Transformation Into BUTA

Using Loops

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$$S = \left\{ (p,q) \;\middle|\; \begin{array}{c} \exists \theta \in \mathbb{S} : \\ \exists (p_\theta, q_\theta) \in S_\theta \end{array} \text{ st. } \begin{array}{c} \langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta \\ \text{and} \\ \langle \boldsymbol{\sigma_\theta}, q_\theta, \theta \to \uparrow, q \rangle \in \Delta \end{array} \right\}$$

The son's symbol is needed to close the end of the loop.

Transformation Into BUTA

Using Loops

Important Remark

In the construction, sets of loops **cannot** be considered independently from the symbol they are rooted in.

Counter-Example

Consider $a, b \in \Sigma_0, f \in \Sigma_2$, with only the transitions

$$\langle \{ a, b \}, p, \theta \to \circlearrowleft, q \rangle \cup \{ \langle b, q, \theta \to \uparrow, s' \rangle, \langle f, s, \tau \to \chi(\theta), p \rangle \} \subseteq \Delta$$

Then $\mho^{\theta}(a) = \mho^{\theta}(b) = \{(p, q)\}^*$, but $\mho^{\tau}(f(a, a)) \neq \mho^{\tau}(f(b, b))$.

Thus the loops-based construction has $\Sigma \times \mathbb{T} \times 2^{Q^2}$ instead of only $\mathbb{T} \times 2^{Q^2}$ states (storing the symbol).

Tree Overloops: slight alteration of the notion of *tree loop*.

Advantages wrt. Transformation Into Buta

- Straightforward $\mathbb{T} \times 2^{Q^2}$ instead of $\Sigma \times \mathbb{T} \times 2^{Q^2}$ states
- DTWA $\mathcal{A}$: smaller $|\mathbb{T}| \cdot 2^{|Q| \log_2(|Q|+1)}$ upper bound on states
- 2 to 100 times smaller BUTA in average (random tests)

$(p, q) \in Q^2$ is an **overloop** of $\mathcal{A}$ on $t|_\alpha$ if there exists a run which starts in p, ends in q at the *parent* of the root α, and always stays in the subtree, except for the last configuration.

Parent of ε is $\overline{\varepsilon}$. A TWA $\mathcal{A}$ must be *escaped* into $\mathcal{A}' = \langle\, \Sigma,\ Q \uplus \{\checkmark\},\ I, F,\ \Delta \uplus \langle \Sigma, F, \star \to \uparrow, \checkmark \rangle \,\rangle$.

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Idea: Compute loops, then check for $\uparrow$-transitions

Definition: Up-Closure

Let $L \subseteq Q^2, \tau \in \mathbb{T}$ and $\sigma \in \Sigma$:

$$\mathcal{U}_\sigma^\tau[L] \stackrel{\text{def}}{=} \{ (p,q) \mid \exists p' : (p,p') \in L \text{ and } \langle \sigma, p', \tau \to \uparrow, q \rangle \in \Delta \} \quad .$$

Theorem: Up-Closure

Let $\mathcal{A}$ be a TWA. If L is the set of all loops of $\mathcal{A}$ on a subtree $u = t|_\alpha$, then $\mathcal{U}_{t(\alpha)}^{\natural\alpha}[L]$ is the set of all overloops of $\mathcal{A}$ on u.

Overloops on Leaves

Let $a \in \Sigma_0$, we have $\mathfrak{O}^\tau(a) = \mathcal{U}_a^\tau[\mathcal{H}_a^{\tau *}]$.

Overloops on Inner Nodes

Let $f \in \Sigma_2$, and $u = f(u_{\boldsymbol{0}}, u_{\boldsymbol{1}})$; root of type τ. To build a *loop* (p, q_θ) on a subtree, we need to...

1. choose a side: $\theta \in \mathbb{S} \stackrel{\text{def}}{=} \{\, \boldsymbol{0}, \boldsymbol{1} \,\}$
2. find an existing overloop on that side: $(p_\theta, q_\theta) \in \mathfrak{O}^\theta(u_\theta)$
3. such that one can connect the beginning:
   1. $\langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta^a$
   2. *unlike loops, the end is already conected!*

---

[a] $\chi(\cdot) : \mathbb{S} \to \{\swarrow, \searrow\}$ such that $\chi(\boldsymbol{0}) = \swarrow$ and $\chi(\boldsymbol{1}) = \searrow$.

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Overloops on Inner Nodes

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$$\mathcal{U}_f^{\tau}\left[\left(\mathcal{H}_f^{\tau} \cup \left\{ (p, q_\theta) \;\middle|\; \begin{array}{l} \exists \theta \in \mathbb{S} : \\ \exists p_\theta \in Q \end{array} : \begin{array}{l} \langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta \\ \text{and } (p_\theta, q_\theta) \in \overleftarrow{\mathfrak{O}}^{\theta}(u_\theta) \end{array} \right\}\right)^*\right]$$

Transformation Into BUTA Using Overloops

Almost the same the the loops-based version, but:

1. We compute sets of *overloops* instead of *loops*
2. Symbols are not stored in the states (not needed): thus $\Sigma \times \mathbb{T} \times 2^{Q^2}$ becomes $\mathbb{T} \times 2^{Q^2}$ again.
3. Final states are

$$\{ (\star, O) \in States \mid O \cap (I \times \{\checkmark\}) \neq \varnothing \}$$

Acceptance criterion: Final Loops & Overloops

A term t is accepted iff

- $\mho^\tau(t) \cap I \times F \neq \varnothing$
- $\overline{\mho}^\tau(t) \cap I \times \{\checkmark\} \neq \varnothing$ (because of $\mathcal{A}' = \langle\ \Sigma,\ Q \uplus \{\checkmark\},\ I, F,\ \Delta \uplus \langle \Sigma, F, \star \to \uparrow, \checkmark \rangle\ \rangle$)

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Deterministic TWA: Definition

A TWA $\mathcal{A} = \langle \Sigma, Q, I, F, \Delta \rangle$ is **deterministic** (ie. a **DTWA**) if[a] for all $\sigma \in \Sigma, p \in Q, \tau \in \mathbb{T}$, $|\langle \sigma, p, \tau \to \mathbb{M}, Q \rangle \cap \Delta| \leqslant 1$.

[a] We do not need the usual, stronger definition, where I is a singleton.

Theorem: Deterministic Upper-Bound

In general, the overloops-based BUTA $\mathcal{B}$ has $|\mathbb{T}| \times 2^{|Q|^2}$ states. However, it has at most $|\mathbb{T}| \cdot 2^{|Q| \log_2(|Q|+1)}$ states if $\mathcal{A}$ is a DTWA.

Proof Idea (Full Proof in Appendix)

Sets of overloops on a given subterm are functional (ie. right-unique). Each computed state stores the set of overloops on a given subterm; thus there are at most $|Q+1|^{|Q|}$ of them, as opposed to $2^{|Q|^2}$ in the general case.

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Testing emptiness of a TWA is an ExpTime-complete problem.
Practical problems:

- XML Queries
- Satisfiability of some XPath fragments
- But also model-checking...

Standard approach: tranform into BUTA, then test emptiness. We propose another approach:

- An "over-approximation"; *may* detect emptiness
- Executes in polynomial time and space
- Very (surprisingly) accurate in our random tests

Over-Approximation of the Emptiness Problem, Using Overloops

0. **Input:** An *escaped* TWA $\mathcal{A} = \langle \Sigma, Q, I, F, \Delta \rangle$
1. **Initialise** $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_\star$ to $\varnothing$
2. **for each** $a \in \Sigma_0, \tau \in \mathbb{T}$ **do**
   - $\mathcal{L}_\tau \leftarrow \mathcal{L}_\tau \cup \mathcal{U}_a^\tau[\mathcal{H}_a^{\tau *}]$
3. **repeat until** $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_\star$ remain unchanged
   - **for each** $f \in \Sigma_2, \tau \in \mathbb{T}$ **do**
     - $\mathcal{L}_\tau \leftarrow \mathcal{L}_\tau \cup \mathcal{U}_f^\tau[(\mathcal{H}_f^\tau \cup S)^*]$
       with S the set of simple loops built on $\mathcal{L}_0$ and $\mathcal{L}_1$.
4. **Output:** *Empty* if $\mathcal{L}_\star \cap (I \times \{\checkmark\}) = \varnothing$, else *Unknown*

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$$S = \left\{ (p, q_\theta) \;\middle|\; \begin{array}{l} \exists \theta \in \mathbb{S} : \\ \exists p_\theta \in Q \end{array} \;\middle|\; \begin{array}{l} \langle f, p, \tau \to \chi(\theta), p_\theta \rangle \in \Delta \\ \text{and } (p_\theta, q_\theta) \in \mathcal{L}_\theta \end{array} \right\}$$

- Approach can be made coarser or finer: from a variant with no type information (ie. $\mathcal{L} = \mathcal{L}_{\boldsymbol{0}} \cup \mathcal{L}_{\boldsymbol{1}} \cup \mathcal{L}_{\star}$) to someting equivalent to transformation to BUTA
- The presented variant is polynomial in time and space
- Astonishing accuracy in random tests: Out of $\approx 20\,000$ TWA ($2 \leqslant |Q| \leqslant 20$), 75% of which had empty languages, only two *Unknown* instead of *Empty*.

Caveat

Our generation scheme was simplistic. Trivial instances? More tests to be made using uniform generation scheme.

Conclusion and Perspectives

What We Have Seen

- Two TWA Membership algorithms (loops & overloops)
- Two transformations from TWA into BUTA (idem)
- Overloops-based BUTA have expected states
- Overloops-based BUTA smaller (2 to 100 times...)
- DTWA: Lower upper-bound on BUTA states with overloops
- Approximation: polynomial, accurate in random tests

What Is Left To Do

- Test approximation with uniform DTWA generation scheme
- Characterise classes of TWA using overloops (det. etc)
- Significant size reductions on TWA (optimise queries etc)

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Transformation Into BUTA Using Overloops

0. **Input:** An *escaped* TWA $\mathcal{A} = \langle \Sigma, Q, I, F, \Delta \rangle$
1. **Initialise** *States* and *Rules* to $\varnothing$
2. **for each** $a \in \Sigma_0, \tau \in \mathbb{T}$ **do**
   - let $P = (\tau, \mathcal{U}_a^\tau[\mathcal{H}_a^{\tau *}])$
     add $a \to P$ to *Rules* and P to *States*
3. **repeat until** *Rules* remain unchanged
   - **for each** $f \in \Sigma_2, \tau \in \mathbb{T}$ **do**
     - add every $f(P_0, P_1) \to P$ to *Rules* and P to *States* where $P_0, P_1 \in States$ such that $P_0 = (\mathbf{0}, S_0)$ and $P_1 = (\mathbf{1}, S_1)$ and $P = (\tau, \mathcal{U}_f^\tau[(\mathcal{H}_f^\tau \cup S)^*])$, with S the set of simple loops built on the sons.
4. **Output:** A BUTA $\mathcal{B}$ equivalent to $\mathcal{A}$: $\mathcal{B} = \langle \Sigma, States, \{ (\star, O) \in States \mid O \cap (I \times \{\checkmark\}) \neq \varnothing \}, Rules \rangle$

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No need to store the son's symbol anywhere.

Transformation Into BUTA in the Deterministic Case

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A relation $R \subseteq Q^2$ is **functional** (or **right-unique**, or **a partial function**) if, for all $p, q, q' \in Q$, pRq and $pRq' \implies q = q'$.

There are $2^{|Q|^2}$ binary relations on Q, of which $|Q+1|^{|Q|}$ are partial functions, of which $|Q|^{|Q|}$ are total functions.

If a relation R is functional, then so is R^k, for any $k \in \mathbb{N}$.

[a] We do not need the usual, stronger definition, where I is a singleton.

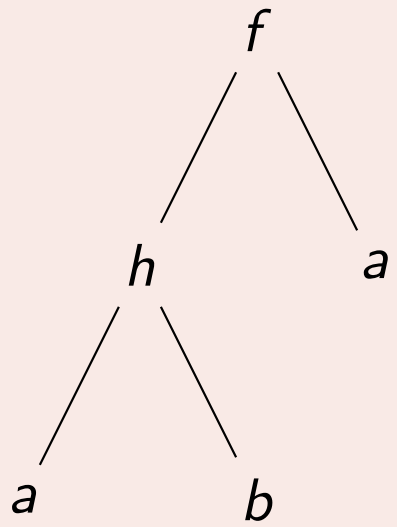
Transformation Into BUTA in the Deterministic Case

Idea: Each state built is the set of overloops (resp. loops) on some tree. We show that the set of overloops on a tree is functional.

Lemma: Deterministic TWA

If $\mathcal{A}$ is a deterministic TWA, then $\twoheadrightarrow_{\mathcal{A}}$ is functional.

Not enough to make sets of *loops* functional:



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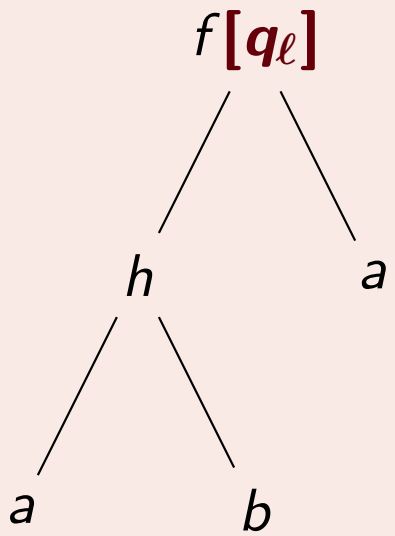

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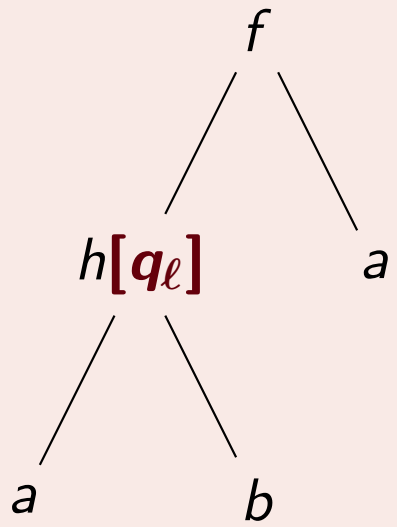

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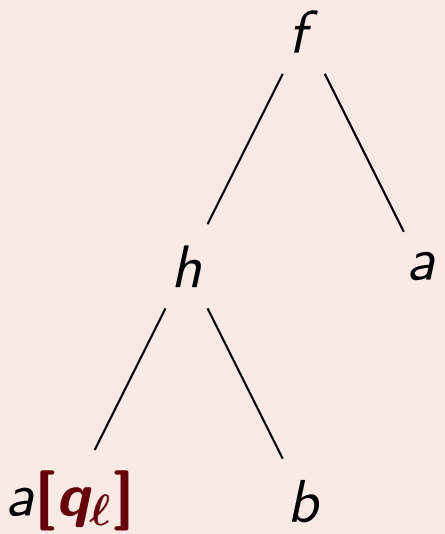

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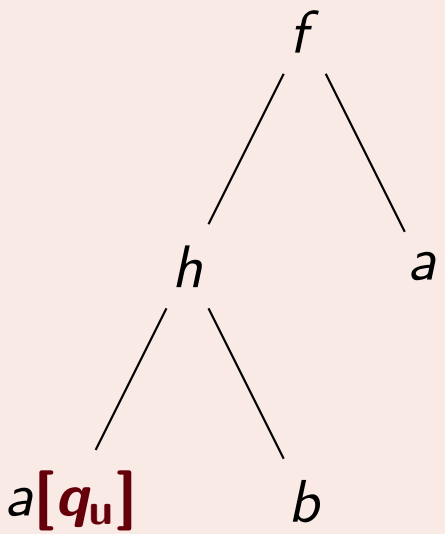

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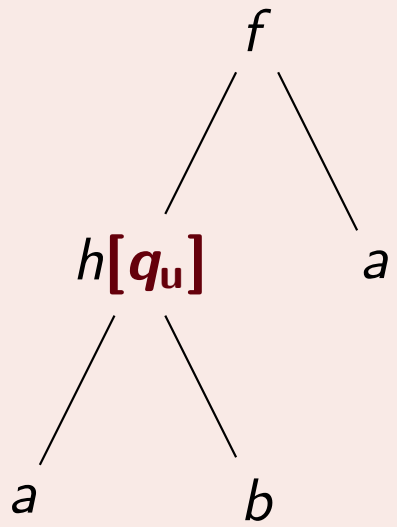

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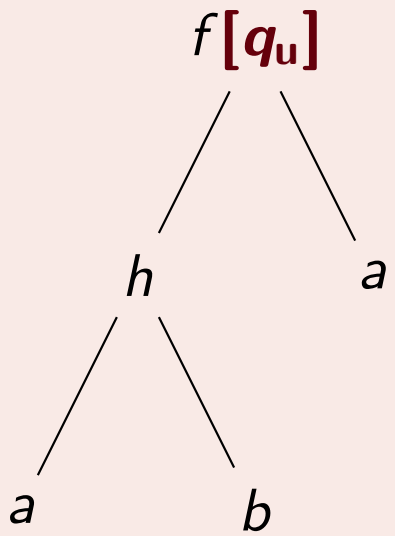

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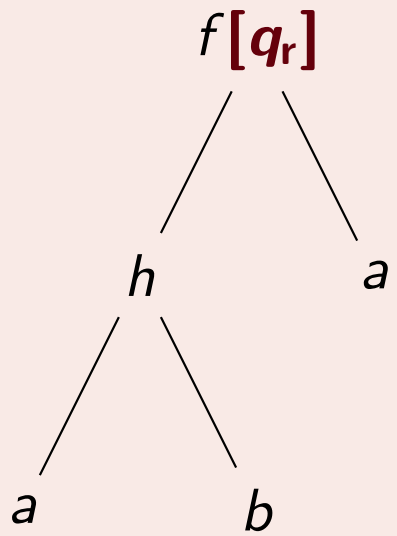

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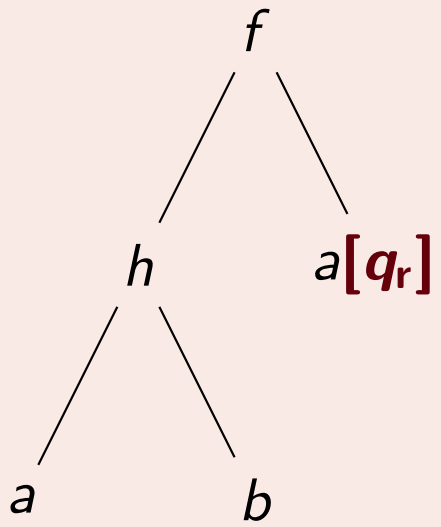

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If $\mathcal{A}$ is a deterministic TWA, then $\twoheadrightarrow_{\mathcal{A}}$ is functional.

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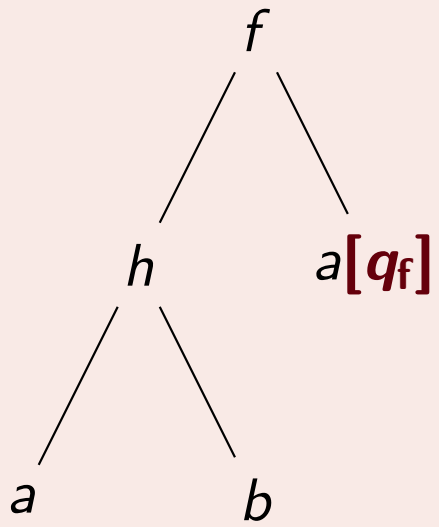

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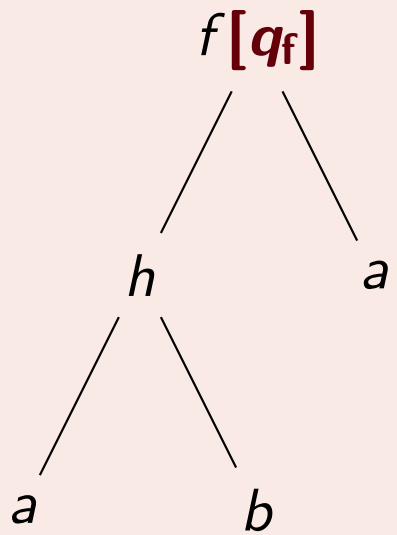

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Loops on this tree: $\{(q_\ell, q_\ell), (q_\ell, q_u), (q_\ell, q_f), \dots\}$. Not functional.

Transformation Into BUTA in the Deterministic Case

Lemma: Hidden Loops

Let $p, q, q' \in Q$, $q \neq q'$ such that (p, q) and (p, q') are loops of a TWA $\mathcal{A}$ on a given subtree $t|_\alpha$. Then if $\mathcal{A}$ is deterministic, either (q, q') or (q', q) must be a loop of $\mathcal{A}$ on $t|_\alpha$.

By definition, there exist two runs $c_0, \ldots, c_n$ and $d_0, \ldots, d_m$ such that $c_0 = d_0 = (\alpha, p)$, $c_n = (\alpha, q)$ and $d_m = (\alpha, q')$.

- If $n = m$ then $c_0 \twoheadrightarrow^n c_n$ and $c_0 \twoheadrightarrow^n d_n$. It follows that $c_n = d_m$. But this contradicts $q \neq q'$, so we must have $n \neq m$.
- Say that $n < m$. Then $c_n = d_n$, and $(\alpha, q) = d_n, \ldots, d_m = (\alpha, q')$ forms a run. Therefore (q, q') is a loop.
- If $n > m$, then by the same arguments (q', q) is a loop.

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Let $p, q, q' \in Q$, such that (p, q) and (p, q') are overloops of a TWA $\mathcal{A}$ on a given subtree $t|_\alpha$. Then if $\mathcal{A}$ is deterministic, $q = q'$.

We have two runs

- $(\alpha, p), \ldots, (\alpha, s), (\overline{\mathfrak{p}}(\alpha), q)$
- $(\alpha, p), \ldots, (\alpha, s'), (\overline{\mathfrak{p}}(\alpha), q')$

Thus (p, s) and (p, s') are loops.

- If $s = s'$, then $q = q'$, because $\twoheadrightarrow$ is functional.
- If $s \neq s'$, then say (s, s') is a loop. So there exist $s_1, \ldots, s_n \in Q, \beta_1 \trianglelefteq \alpha, \ldots, \beta_n \trianglelefteq \alpha$ such that $(\alpha, s), (\beta_1, s_1), \ldots, (\beta_n, s_n), (\alpha, s')$ is a run. Thus $(\alpha, s) \twoheadrightarrow (\overline{\mathfrak{p}}(\alpha), q)$ and $(\alpha, s) \twoheadrightarrow (\beta_1, s_1)$. It follows that $\overline{\mathfrak{p}}(\alpha) = \beta_1 \trianglelefteq \alpha$: contradiction.

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Transformation Into BUTA in the Deterministic Case

Theorem: Deterministic Upper-Bound

In general, the overloops-based BUTA $\mathcal{B}$ has $|\mathbb{T}| \times 2^{|Q|^2}$ states. However, it has at most $|\mathbb{T}| \cdot 2^{|Q| \log_2(|Q|+1)}$ states if $\mathcal{A}$ is a DTWA.

By construction, for every state $P = (\tau, L)$ generated for $\mathcal{B}$ by the overloops-based algorithm, there exists at least a subtree t such that L is the set of overloops of $\mathcal{A}$ on t. Thus, by the previous lemma, L is functional. Therefore, there are at most $|\mathbb{T}| \cdot |Q+1|^{|Q|}$ states (or, equivalently, $|\mathbb{T}| \cdot 2^{|Q| \log_2(|Q|+1)}$).

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